Spherical Harmonics, by Atkinson and Han

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Section 1.3: Basic Results Related to the Sphere

Page 7, equation (1.15)

Consider the function

$$\begin{aligned} \Psi : [-1,1] \times \mathbb{S}^{d-2} &\to \mathbb{S}^{d-1} \\ (t,\xi) &\to te_d + \sqrt{1-t^2}\xi. \end{aligned}$$

Here, $\xi \in \mathbb{S}^{d-2} \subset \mathbb{R}^{d-1}$, and we are identifying ξ with $\xi + 0e_d \in \mathbb{R}^d$. The surface area element dS^{d-1} can be computed in terms of dt and dS^{d-2}, as follows. For fixed t, the function Ψ maps the $(d-2)$-dimensional manifold $\{t\} \times \mathbb{S}^{d-2}$ to a translate at height t of its dilation by $\sqrt{1-t^2}$. Thus, the distorsion factor is $(1-t^2)^{(d-2)/2}$. On the other hand, for fixed ξ, the function Ψ maps $[-1,1] \times \{\xi\}$ to the arc $te_d + \sqrt{1-t^2}\xi$, and $\|\nabla_t \Psi\| = \frac{1}{\sqrt{1-t^2}}$. By orthogonality, dS^{d-1} is the product of $(1-t^2)^{(d-2)/2} dS^{d-2}$ and $\frac{1}{\sqrt{1-t^2}}dt$, obtaining

$$dS^{d-1} = (1-t^2)^{(d-3)/2} dS^{d-2} dt.$$

Because of this relation, we can identify the space of integrable functions on $\mathbb{S}^{d-1}$ which are rotationally invariant with respect to e_d with the space

$$L^1_{\frac{d-3}{2}}(-1,1) = \{f : (-1,1) \to \mathbb{C} : \int_{-1}^{1} |f(t)|(1-t^2)^{(d-3)/2} dt < \infty\}. \tag{1}$$

Spherical Harmonics - Definition and Properties

The goal of Chapter 2 is to define the space of spherical harmonics as the **only irreducible** system of **invariant** function spaces on the sphere that is **complete and closed**. Here, invariant means under orthogonal transformations, and irreducible means that it cannot be written as a direct sum of two invariant subspaces .

The starting point is the space $\mathbb{H}_n^d$ of **homogeneous polynomials of degree n in dimension d**, which has dimension $\binom{n+d-1}{n}$ and is invariant under orthogonal transformations.

$\mathbb{H}_n^d$ has a proper invariant subspace, $|\cdot|^2 \mathbb{H}_{n-2}^d$, which means $\mathbb{H}_n^d$ is reducible (in particular, the restriction $\mathbb{H}_n^d|_{S^{d-1}}$ is reducible too). What is the orthogonal subspace to $|\cdot|^2 \mathbb{H}_{n-2}^d$? The inner product on $\mathbb{H}_n^d$ is defined by

$$(H_{n,1}, H_{n,2})_{\mathbb{H}_n^d} := H_{n,1}(\nabla)\overline{H_{n,2}(x)}. \tag{2}$$

If P belongs to the invariant subspace $|\cdot|^2\mathbb{H}^d_{n-2}$, then $P(\nabla)$ is the composition of a Laplacian with an $(n-2)$-differential operator. Thus, **harmonic polynomials** are orthogonal to our subspace, and we define $\mathbb{Y}_n(\mathbb{R}^d)$ to be the space of real harmonic homogeneous polynomials of degree n. Observe that $\mathbb{Y}_0(\mathbb{R}^d) = \mathbb{H}^d_0$ and $\mathbb{Y}_1(\mathbb{R}^d) = \mathbb{H}^d_1$.

The restriction of $\mathbb{Y}_n(\mathbb{R}^d)$ to $\mathbb{S}^{d-1}$ is denoted by $\mathbb{Y}^d_n$, and called the **spherical harmonic space** of order n in d dimensions. Both spaces have dimension

$$N_{n,d} = \frac{(2n+d-2)(n+d-3)!}{n!(d-2)!}.$$

Properties of the spaces $\mathbb{H}^d_n$ (see Section 2.4):

1. The Laplacian Δ is surjective from $\mathbb{H}^d_n$ to $\mathbb{H}^d_{n-2}$.
2. For $n \geq 2$, $\mathbb{H}^d_n = \mathbb{Y}^d_n \oplus |\cdot|^2\mathbb{H}^d_{n-2}$, with respect to the inner product (2).
3. $\mathbb{H}^d_n = \mathbb{Y}^d_n \oplus |\cdot|^2\mathbb{Y}^d_{n-2} \oplus |\cdot|^4\mathbb{Y}^d_{n-4} \oplus \cdots \oplus |\cdot|^{2[\frac{n}{2}]}\mathbb{Y}^d_{n-2[\frac{n}{2}]}$.
4. The restriction of any polynomial to S^{d-1} is a sum of spherical harmonics, since

$$\left(\sum_{j=0}^{n} \mathbb{H}^d_j\right)\Bigg|_{\mathbb{S}^{d-1}} = \sum_{j=0}^{n} \mathbb{Y}^d_j.$$

5. A polynomial $H_n \in \mathbb{H}^d_n$ is harmonic if and only if

$$\int_{\mathbb{S}^{d-1}} H_n(\xi)\overline{H_{n-2}(\xi)}dS^{d-1}(\xi) = 0 \ \ \forall H_{n-2} \in \mathbb{H}^d_{n-2}.$$

6. A procedure to decompose a given $H_n \in \mathbb{H}^d_n$ into harmonics is given on pages 33-34.

Invariant functions on $\mathbb{Y}^d_n$ and Legendre Polynomials

Denoting by $\mathbb{O}^d(\xi)$ the subgroup of orthogonal transformations in S^{d-1} which fix $\xi \in S^{d-1}$, we have (**Theorem 2.8**) that a spherical harmonic $Y_n \in \mathbb{Y}^d_n$ is invariant with respect to $\mathbb{O}^d(\xi)$ if and only if

$$Y_n(\eta) = Y_n(\xi)P_{n,d}(\xi \cdot \eta), \quad \forall \eta \in \mathbb{S}^{d-1}, \tag{3}$$

where $P_{n,d}$ is the Legendre polynomial of degree n in dimension d (see 2.19). Some of its properties include $P_{n,d}(1) = 1$, $P_{n,d}(t) \leq 1$ for $t \in [-1,1]$, and for fixed $\xi \in \mathbb{S}^{d-1}$,

$$\int_{\mathbb{S}^{d-1}} |P_{n,d}(\xi \cdot \eta)|^2 d\sigma(\eta) = \frac{|\mathbb{S}^{d-1}|}{N_{n,d}}.$$

These polynomials play a crucial role in the theory.

Theorem 1. Addition Theorem (2.9) *If $\{Y_{n,j}\}$ is an orthonormal basis of $\mathbb{Y}_n^d$, then*

$$\sum_{j=1}^{N_{n,d}} Y_{n,j}(\xi)\overline{Y_{n,j}(\eta)} = \frac{N_{n,d}}{|\mathbb{S}^{d-1}|} P_{n,d}(\xi \cdot \eta) =: K_{n,d}(\xi, \eta), \quad \forall \xi, \eta \in \mathbb{S}^{d-1}.$$

Consequences of the Addition Theorem include:

1. $K_{n,d}(\xi, \eta)$ is the reproducing kernel of $\mathbb{Y}_n^d$, *i.e.* for every $Y_n \in \mathbb{Y}_n^d$ and $\xi \in \mathbb{S}^{d-1}$,

$$Y_n(\xi) = (Y_n, K_{n,d}(\xi, \cdot)) .$$

2. $\mathbb{Y}_n^d$ is irreducible.

3. $\|Y_n\|_{L^\infty(\mathbb{S}^{d-1})} \leq \left(\frac{N_{n,d}}{|\mathbb{S}^{d-1}|}\right)^{1/2} \|Y_n\|_{L^2(\mathbb{S}^{d-1})}$.
   Compare this to the trivial bound $\|Y_n\|_{L^2(\mathbb{S}^{d-1})} \leq |\mathbb{S}^{d-1}|^{1/2} \|Y_n\|_{L^\infty(\mathbb{S}^{d-1})}$.

The space of all spherical harmonics of order less than or equal to m is

$$\mathbb{Y}_{0:m}^d := \bigoplus_{n=0}^{m} \mathbb{Y}_n^d,$$

and its reproducing kernel is

$$K_{0:m,d}(\xi, \eta) = \frac{1}{|\mathbb{S}^{d-1}|} \sum_{n=0}^{m} N_{n,d} P_{n,d}(\xi \cdot \eta).$$

The Projection Operator

Given an orthonormal basis $\{Y_{n,j}\}$ of $\mathbb{Y}_n^d$, then the projection of $f \in L^2(S^{d-1})$ on $\mathbb{Y}_n^d$ can be written, using the Addition Theorem, as

$$\left(\mathcal{P}_{n,d} f\right)(\xi) = \frac{N_{n,d}}{|S^{d-1}|} \int_{\mathbb{S}^{d-1}} P_{n,d}(\xi \cdot \eta) f(\eta) d\sigma(\eta). \tag{4}$$

This dispenses us from a concrete choice of basis to compute projections on $\mathbb{Y}_n^d$. In addition, now the projection is well defined for f in the larger space $L^1(S^{d-1})$. Some bounds for the operator $\mathcal{P}_{n,d}$:

- $\|\mathcal{P}_{n,d} f\|_{C(\mathbb{S}^{d-1})} \leq \frac{N_{n,d}}{|S^{d-1}|} \|f\|_{L^1(\mathbb{S}^{d-1})}$.
- $\|\mathcal{P}_{n,d} f\|_{L^1(\mathbb{S}^{d-1})} \leq N_{n,d} \|f\|_{L^1(\mathbb{S}^{d-1})}$.
- $\|\mathcal{P}_{n,d} f\|_{C(\mathbb{S}^{d-1})} \leq \left(\frac{N_{n,d}}{|\mathbb{S}^{d-1}|}\right)^{1/2} \|f\|_{L^2(\mathbb{S}^{d-1})}$.
- $\|\mathcal{P}_{n,d} f\|_{C(\mathbb{S}^{d-1})} \leq N_{n,d}^{1/2} \|f\|_{C(\mathbb{S}^{d-1})}$.

- $\|\mathcal{P}_{n,d} f\|_{L^2(\mathbb{S}^{d-1})} \leq \|f\|_{L^2(\mathbb{S}^{d-1})}$.

Additional properties of the projection operator include:

1. The projection operator and orthogonal transformations commute,
$$\mathcal{P}_{n,d}(f_A) = (\mathcal{P}_{n,d} f)_A \quad \forall A \in \mathbb{O}^d.$$
2. If $\mathbb{V}$ is an invariant space, then $\mathcal{P}_{n,d}(\mathbb{V})$ is an invariant subspace of $\mathbb{Y}_n^d$.
3. Since $\mathbb{Y}_n^d$ is irreducible, it cannot have proper invariant subspaces. Thus, for any invariant $\mathbb{V}$, either $\mathbb{V} \perp \mathbb{Y}_n^d$ or $\mathcal{P}_{n,d}(\mathbb{V}) = \mathbb{Y}_n^d$ (here $\perp$ is with respect to the $L^2(\mathbb{S}^{d-1})$ inner product).
4. Letting $\mathbb{V} = \mathbb{Y}_m^d$, for $m \neq n$ we have $\mathbb{Y}_m^d \perp \mathbb{Y}_n^d$.

Combining the Addition Theorem and the definition of the projection operator, we obtain:

The Funk-Hecke Formula. Let $f \in L^1_{\frac{d-3}{2}}(-1,1)$, $\xi \in \mathbb{S}^{d-1}$ and $Y_n \in \mathbb{Y}_n^d$. Then
$$\int_{\mathbb{S}^{d-1}} f(\xi \cdot \eta)\, Y_n(\eta)\, dS^{d-1}(\eta) = \lambda_n\, Y_n(\xi),$$
where
$$\lambda_n = |\mathbb{S}^{d-2}| \int_{-1}^{1} P_{n,d}(t)\, f(t)\, (1-t^2)^{(d-3)/2} dt.$$