De Branges transform and the Hilbert inequality

A problem due to H. Montgomery from 1974 asks for the optimal constant c such that

$$\left| \sum_{\substack{m,n=1 \\ m \neq n}}^{N} \frac{a_m \bar{a}_n}{\lambda_n - \lambda_m} \right| \leq c \sum_{n=1}^{N} \frac{|a_n|^2}{\delta_n}$$

where the λ_i are distinct and real, the a_n are complex, and $\delta_n = \min\{|\lambda_n - \lambda_m| : m \neq n\}$. (It is conjectured that $c = \pi$; this is known if all δ_n are equal.)

In 2004 X.J. Li proposed a trigonometric interpolation problem whose solution would imply that $c = \pi$. In this talk I will describe Li's approach using the Weyl transform of a canonical system and discuss feasibility of solving it. The interpolation problem is formulated in a Hilbert space of polynomials, while the norm of the interpolating function is evaluated in the solution space of an ODE of the form

$$JY_x = zH(x)Y,$$

where $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $H(x)$ is a symmetric matrix. The talk is intended to be accessible to graduate students with some knowledge of complex analysis and Hilbert spaces.