Legendre Polynomials: Properties

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Section 2.7.1

Proposition 1. *If $f \in C^n([-1,1])$, then*

$$\int_{-1}^{1} f(t) P_{n,d}(t)(1-t^2)^{\frac{d-3}{2}}\,dt = R_{n,d}\int_{-1}^{1} f^{(n)}(t)(1-t^2)^{n+\frac{d-3}{2}}\,dt$$

where

$$R_{n,d} = \frac{\Gamma(\frac{d-1}{2})}{2^n\Gamma(n+\frac{d-1}{2})}$$

Cool Equation 2.

$$\int_{-1}^{1}[P_{n,d}(t)]^2(1-t^2)^{\frac{d-3}{2}}\,dt = \frac{\sqrt{\pi}\Gamma(\frac{d-1}{2})}{N_{n,d}\Gamma(\frac{d}{2})}$$

In particular, for $d = 3$, we have $N_{n,3} = 2n+1$, and

$$\int_{-1}^{1}[P_{n,3}(t)]^2\,dt = \frac{2}{2n+1}$$

For $d = 2$, $N_{n,2} = 2$, and

$$\int_{-1}^{1}[P_{n,2}(t)]^2(1-t^2)^{-1/2}\,dt = \frac{\pi}{2}$$

Section 2.7.2

Definition 3. We define

$$L_d g(t) \stackrel{\text{def}}{=} (1-t^2)^{\frac{3-d}{2}}\frac{d}{dt}\left[(1-t^2)^{\frac{d-1}{2}}\frac{d}{dt}g(t)\right], \qquad g \in C^2[-1,1]$$

Alternatively,

$$L_d g(t) = (1-t^2)g''(t) - (d-1)tg'(t) \tag{1}$$

◁

Definition 4. We define a weighted inner product $(\cdot,\cdot)_d$ by

$$(f,g)_d \stackrel{\text{def}}{=} \int_{-1}^{1} f(t)g(t)(1-t^2)^{\frac{d-3}{2}}\,dt \qquad ◁$$

Cool Equation 5. By integration by parts, we find that for all $f, g \in C^2[-1,1]$

$$(L_d f, g)_d = (f, L_d g)_d$$

Note: This concept can be generalized into a rather neat method of constructing self-adjoint operators, which is detailed in the theorem below.

Theorem 6. *Let μ be a measure on $[a,b] \subseteq \mathbb{R}$ which is absolutely continuous with respect to the Lebesgue measure and for which $R(t) \stackrel{def}{=} \frac{d\mu}{dt} > 0$. Suppose we define a μ-inner product on $C^2[a,b]$ by*

$$\langle f, g \rangle_\mu = \int_a^b f(t)g(t)\; d\mu(t) = \int_a^b f(t)g(t)R(t)\; dt$$

Then for $h \in C^1[a,b]$ with $p(a) = p(b) = 0$, the differential operator

$$L_h f = \frac{1}{R(t)} \frac{d}{dt} \left(h(t) \frac{d}{dt} f(t) \right)$$

is self-adjoint with respect to the μ-inner product. That is,

$$\langle L_h f, g \rangle_\mu = \langle f, L_h g \rangle_\mu$$

Proof.

$$\begin{aligned} \langle L_h f, g \rangle_\mu &= \int_a^b L_h f(t) g(t) R(t)\; dt \\ &= \int_a^b \frac{1}{R(t)} \frac{d}{dt} \left(h(t) f'(t) \right) g(t) R(t)\; dt \\ &= \int_a^b \frac{d}{dt} \left(h(t) f'(t) \right) g(t)\; dt \end{aligned}$$

Let us integrate by parts:

$$\begin{aligned} &= h(t)f'(t)g(t)|_a^b - \int_a^b h(t)f'(t)g'(t)\; dt \\ &= h(b)f'(b)g(b) - h(a)f'(a)g(a) - \int_a^b h(t)f'(t)g'(t)\; dt \\ &= -\int_a^b h(t)f'(t)g'(t)\; dt \end{aligned}$$

By much the same work, we see that $\langle L_h g, f \rangle_\mu = -\int_a^b h(t)f'(t)g'(t)\; dt$ also. The result follows by the obvious fact that the μ-inner product is symmetric.

$$\langle L_h f, g \rangle_\mu = \langle L_h g, f \rangle_\mu = \langle f, L_h g \rangle_\mu$$

□

Cool Equation 7. The L_d operator maps polynomials of degree n to polynomials of degree n, as seen in (1). Since $(p_m, P_{n,d})_d = 0$ for any polynomial of degree $m < n$ (see (2.69) in the text) we have

$$(P_{n,d}, L_d P_{m,d})_d = (P_{m,d}, L_d P_{n,d})_d = 0 \tag{2}$$

Since $\{P_{m,d}\}_{0 \le m \le n}$ form an orthogonal basis for all polynomials of degree less than or equal to n, we may write

$$L_d P_{n,d} = \sum_{m=0}^{n} c_m P_{m,d}$$

Equation (2) tells us that for $0 \le m \le n-1$ we have $c_m = 0$, i.e. $L_d P_{n,d}$ is a multiple of $P_{n,d}(t)$.

We may write

$$P_{n,d}(t) = a_{n,d}^0 t^n + \text{l.d.t.}$$

where l.d.t. denotes the lower degree terms. Then

$$L_d P_{n,d}(t) = -n(n+d-2)a_{n,d}^0 t^n + \text{l.d.t.}$$

Since $L_d P_{n,d}$ is a multiple of $P_{n,d}$, this implies that it must be that

$$L_d P_{n,d}(t) = -n(n+d-2)P_{n,d}(t)$$

In other words, $P_{n,d}$ is an eigenfunction of the differential operator $-L_d$ for the eigenvalue $n(n+d-2)$.

Cool Equation 8. By substituting in the simpler definition of L_d into the equation

$$L_d P_{n,d}(t) + n(n+d-2)P_{n,d}(t) = 0$$

we obtain the relation

$$(1-t^2)P''_{n,d}(t) - (d-1)tP'_{n,d}(t) + n(n+d-2)P_{n,d}(t) = 0$$

and this (according to the text) implies that $P_{n,d}$ has no multiple roots in $(-1,1)$.

Why is this? I don't see why $P_{n,d}$ could not have triple (or higher) roots.

Suppose $P_{n,d}$ has k distinct roots $t_1, \cdots, t_k$ in the interval $(-1,1)$ for $k < n$. Then

$$p_k(t) = (t-t_1)\cdots(t-t_k)$$

and

$$P_{n,d}(t) = p_k(t)q_{n-k}(t)$$

q_{n-k} must be positive at 1 and it has no zeros in $(-1,1)$, so $q_{n-k} > 0$ on all of $(-1,1)$. Then, since $\deg p_k = k < n$,

$$\begin{aligned} 0 &= (P_{n,d}, p_k)_d \\ &= \int_{-1}^{1} P_{n,d}(t)p_k(t)(1-t^2)^{\frac{d-3}{2}}\, dt \\ &= \int_{-1}^{1} q_{n-k}(t)p_k(t)^2(1-t^2)^{\frac{d-3}{2}}\, dt > 0 \end{aligned}$$

This is a contradiction. These results are summarized in the following proposition.

Proposition 9. *The Legendre polynomials $P_{n,d}(t)$ has exactly n distinct roots in $(-1,1)$.*

Section 2.7.3

This is a long section, but its primary result is the following recursive formula for any Legendre polynomial $P_{n,d}$ with $d \geq 2$.

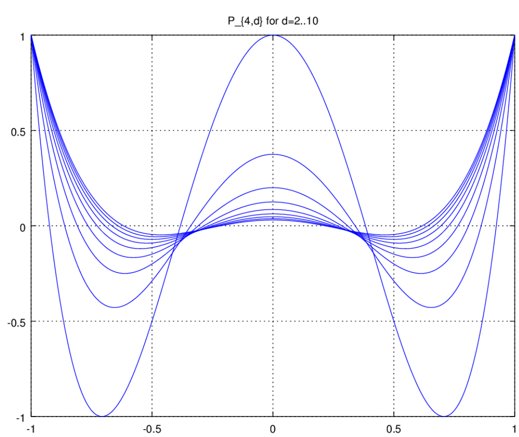
Theorem 10. *Suppose $d \geq 2$.*

$$P_{0,d}(t) = 1 \tag{3}$$

$$P_{1,d}(t) = t \tag{4}$$

$$P_{n,d}(t) = \frac{2n+d-4}{n+d-3}tP_{n-1,d}(t) - \frac{n-1}{n+d-3}P_{n-2,d}(t) \qquad \text{for } n \geq 2 \tag{5}$$

We may use these recursive formulas to compute different $P_{n,d}$. Several examples may be found on page 47. Fixing $n = 4$ and varying d, we obtain cool plots like



Much calculation, which may be found on pages 50-51, yields the following equation.

Cool Equation 11.

$$P'_{n,d}(t) = \frac{n(n+d-2)}{d-1} P_{n-1,d+2}(t), \qquad n \geq 1, \quad d \geq 2$$

More generally, we have the following equality involving higher derivatives.

$$P^{(j)}_{n,d}(t) = \frac{n!\,(n+j+d-3)!\,\Gamma(\frac{d-1}{2})}{2^j (n-j)!\,(n+d-3)!\,\Gamma(j+\frac{d-1}{2})} P_{n-j,d+2j}(t) \qquad n \geq j, \qquad d \geq 2$$

After even more calculation, we arrive at a nice equation.

Theorem 12.

$$(1-t^2)P'_{n,d}(t) = n\,[P_{n-1,d}(t) - tP_{n,d}(t)] \qquad n \geq 1,\ d \geq 2,\ t \in [-1,1]$$

Section 2.7.4