MATH 754: Homework 3 (Due December 10)

1. Let X be a Banach space. Prove that X is reflexive if and only if X^* is reflexive.

2. Let X, Y be normed spaces and $T \in B(X, Y)$. Let us define the **adjoint of T**, $T^* : Y^* \to X^*$ by $T^*(f) = f \circ T$.

   (a) Prove that $T^* \in B(Y^*, X^*)$ and that $\|T^*\| = \|T\|$.

   (b) If we define T^{**} as the adjoint of T^*, then by part (a) we have that $T^{**} \in B(X^{**}, Y^{**})$. If we identify X with its image $J_X(X) \subseteq X^{**}$ and Y with $J_Y(Y) \subseteq Y^{**}$, prove that the restriction of T^{**} to X is equal to T.

   (c) Prove that T^* is injective if and only if $T(X)$ is dense in Y.

   (d) Prove that if $T^*(Y^*)$ is dense in X^*, then T is injective, and that the converse is true if X is a reflexive Banach space.

   Hint: For parts (c) and (d), use the Criterion for density that we saw after the last corollary of Hahn-Banach.

3. Let X, Y be normed spaces with $X \neq \{0\}$. Assume that $B(X, Y)$ is a Banach space with the usual norm $\|T\| = \sup\{|T(x)| : x \in X, \|x\| = 1\}$. Show that then Y is a Banach space.

4. Let X be a Banach space and assume that X^* is separable. Prove that X is separable.

5. Show that $L^1[0, 1]$ is not reflexive. (Use the ideas of the proof that ℓ^1 is not reflexive). Deduce that $L^\infty[0, 1]$ and $C[0, 1]$ are not reflexive.

6. In this problem we will see that if μ is not a sigma-finite mesure, then the correspondence $L^\infty(\Omega, \mathcal{M}, \mu) \to (L^1(\Omega, \mathcal{M}, \mu))^*$ is not surjective.

   Let Ω be any uncountable set, μ the counting measure on $\mathcal{P}(\Omega)$.
   Let $\mathcal{M} = \{E \in \mathcal{P}(\Omega) :\ E$ is countable or E^c is countable $\}$ and $\mu_0 = \mu|_{\mathcal{M}}$. Show that $L^1(\mu)) = L^1(\mu_0)$ (i.e. if f is integrable with respect to μ, its support is countable). Show also that $L^\infty(\mu)) \neq L^\infty(\mu_0)$ and that the dual of $L^1(\mu_0)$ is $L^\infty(\mu)$, not $L^\infty(\mu_0)$.

7. Give an example of a subset of $\mathbb{R}$ that is

   (a) First category and dense

   (b) First category and uncountable

   (c) First category, dense and uncountable

   (d) First category with positive Lebesgue measure

   (e) Second category with Lebesgue measure zero

8. Use Baire's Theorem to prove that there exist functions that are continuous on an interval $[a, b] \subseteq \mathbb{R}$ but are not monotone on any subinterval of $[a, b]$.

9. Use Baire's Theorem to prove that if X is a vector space whose Hamel basis has cardinality $\aleph_0$, then X cannot be complete with any norm.

10. Use the Closed Graph Theorem to prove the following:

    Let X, Y be Banach spaces, and let $T : X \to Y$ be a linear operator such that for every $f \in Y^*$, $f \circ T \in X^*$. Show that T is continuous.

11. Let X be a Banach space. We say that a sequence $\{x_n\}_{n\in\mathbb{N}}$ of elements of X is a **Schauder basis** of X if every $x \in X$ has a unique representation of the form

    $$x = \sum_{j=1}^{\infty} \lambda_j x_j,$$

    where $\lambda_j \in \mathbb{F}$ and the convergence of the series is in the norm of X. We define a family of linear operators P_n by

    $$P_n(x) = \sum_{j=1}^{n} \lambda_j x_j,$$

    and a new norm on the space X by $|||x||| = \sup_{n\in\mathbb{N}} \|P_n(x)\|$.

    (a) Show that $|||\cdot|||$ is a norm.

    (b) Show that X is also complete with respect to this new norm.

    (c) Show that each P_n is continuous and that $\sup_{n\in\mathbb{N}} \|P_n\| < \infty$. This number is called the **constant of the Schauder basis**. (Hint: Use Banach-Steinhaus).

    (d) The collection $\{e_n\}$ of sequences that have a 1 at the nth entry and 0's everywhere else is clearly a Schauder basis of ℓ^p, $1 \leq p \leq \infty$. Find the constant of the basis.

    (e) It is known that, for $1 < p < \infty$, the partial sums $S_N f$ of the Fourier series of $f \in L^p[0, 1]$ converge to f in the L^p norm. Show that this implies that the trigonometric system $\{e^{2\pi i n x}\}_{n\in\mathbb{Z}}$ is a Schauder basis of $L^p[0, 1]$, $1 < p < \infty$, and that $P_N f(x) = S_N f(x)$.