MATH 754: Collected facts about normed spaces

1. **Facts about Hilbert spaces**

   Hilbert spaces have the richest geometry among normed spaces. Their 4 basic properties are:

   (a) **Best approximation.** Every non-empty closed convex subset of a Hilbert space X has a unique element of minimal norm.

   The proof is based on the Parallelogram Law, that characterizes the norm of a Hilbert space.

   (b) **Orthogonality**. The inner product in a Hilbert space X allows us to define that two elements $x, y \in X$ are orthogonal if and only if $\langle x, y\rangle = 0$. The orthogonal of a set M is a closed subspace $M^\perp = \{y \in X : \langle x, y\rangle = 0 \ \forall x \in M\}$.

   If M is a closed subspace of a Hilbert space X, then M is **complemented**, *i.e.* there exists a closed subspace N of X such that $M \cap N = \{0\}$ and $X = M + N$. In fact, $N = M^\perp$. The **orthogonal projections** $P_M : X \to M$ and $P_{M^\perp} : X \to M^\perp$ are linear and continuous, and for every $x \in X$, $\|x\|^2 = \|P_M(x)\|^2 + \|P_{M^\perp}(x)\|^2$.

   (c) **Duality.** Riesz Representation Theorem: Every Hilbert space X is isometrically isomorphic to its dual X^*. The isometry is

   $$\begin{array}{rcl} \mathcal{I} : X & \to & X^* \\ y & \to & \Lambda_y, \end{array}$$

   where $\Lambda_y(x) = \langle x, y\rangle$.

   (d) **Basis.** Every Hilbert space X has an orthonormal basis $\{e_\alpha\}_{\alpha\in A}$, such that for every $x \in X$, x can be written as the orthogonal sum

   $$x = \sum_{\alpha\in A} \langle x, e_\alpha\rangle e_\alpha.$$

   The sum is to be understood in the norm of the space X. As a consequence, every Hilbert space X is isometrically isomorphic to $\ell^2(A)$, where A is the index set of any basis of X. The isometry is the *Fourier transform* $\mathcal{F}$, defined by

   $$\begin{array}{rcl} \mathcal{F} : X & \to & \ell^2(A) \\ x & \to & (\langle x, e_\alpha\rangle)_{\alpha\in A}. \end{array}$$

2. **Extension to Banach spaces**

   Banach spaces do not have in general the above 4 properties.

   (a) **No best approximation.** We have seen examples of closed convex sets in a Banach space in which there is no element of minimal norm, or there are infinitely many of them.

(b) **Orthogonality.** Without an inner product, there is not a natural way to define when two elements of X are orthogonal. However, the concept of **complemented subspace** still exists, and Hahn-Banach and the Closed Graph Thm allows us to define a (non-orthogonal) projection from X to a complemented subspace M.

The only Banach spaces with the property that any closed subspace is completemented are the finite-dimensional ones and the Hilbert spaces. Any other infinite dimensional Banach space contains uncompletemented closed subspaces.

In a Banach space X we can also define the notion of the orthogonal of a set $A \subseteq X$, but $A^{\perp}$ is a subset of X^*, (not X):

$$A^{\perp} = \{\Lambda \in X^* : \Lambda(x) = 0 \text{ for all } x \in A\}.$$

$A^{\perp}$ is a closed subspace of X^*. We will see more about this when we study *weak topologies*.

(c) **Duality.** In general, there is no isometry between a Banach space X and its dual X^*. The main tool to study X^* is the Hahn-Banach theorem, that allows us to extend any linear functional f defined on a subspace M of a normed space X to a linear functional Λ defined on all of X, **without increasing the norm** (i.e. $\|f\| = \|\Lambda\|$).

There exists a natural isometry between X and its bidual X^{**}, given by

$$\begin{aligned} J : X &\to X^{**} \\ x &\to J(x), \end{aligned}$$

where $J(x)(\Lambda) = \Lambda(x)$. However, J is not surjective in general. Those spaces X for which the natural isometry J is surjective are called **reflexive**.

(d) **Basis.** Being a vector space, any Banach space has a Hamel basis, but a Hamel basis does not interact well with the topology. Some Banach spaces have a Schauder basis, a sequence $\{e_n\}_{n\in\mathbb{N}}$ of elements of X such that every $x \in X$ has a unique representation $\sum_{n=1}^{\infty} \alpha_n e_n$, with $\alpha_n \in \mathbb{F}$. The convergence of the series is understood to be in the norm topology.

The spaces $L^p[0,1]$ have a Schauder basis if $1 < p < \infty$ (the trigonometric basis or the Haar basis), and so do ℓ^p, $1 \leq p < \infty$ and c_0. A Banach space with a Schauder basis is necessarily separable, but not every separable Banach space admits a Schauder basis. To study Schauder bases we will need the Principle of Uniform Boundedness.

3. **Facts about finite-dimensional spaces and subspaces**

(a) Let X be a vector space of finite dimension. Then all norms defined on X are equivalent (they define the same topology).

(b) Let M be a finite dimensional subspace of a normed space X. Then M is closed.

(c) Let X be a normed space, M a closed subspace of X and N a finite dimensional subspace of X. Then $M + N$ is closed.

The proof is exactly the same as that of 1b.

(d) Let X, Y be normed spaces and $T : X \to Y$ a linear operator.

i. If $\dim(X)$ is finite, then T is continuous.

ii. If $\dim(Y)$ is finite, then T is continuous if and only if its kernel is closed.
The proof of 1d(ii) uses quotient spaces.

iii. If X, Y are Banach spaces and $\dim(X) = \infty$, then there exists a linear operator $S : X \to Y$ that is not continuous.

(e) If $dim(X)$ is finite, every closed subspace M of X is complemented.

The proof uses Hahn-Banach. The projection on M is continuous by the Closed Graph Theorem

4. **Facts about completeness of normed spaces**

(a) Every closed subspace of a Banach space is complete.

(b) If Y is a Banach space, then $B(X, Y)$ is a Banach space. Note that X does not need to be complete.

(c) In particular, for any normed space X, its dual X^* is complete.

(d) The converse of 2b: Let X, Y be normed spaces, with $X \neq \{0\}$, and assume that $B(X, Y)$ is Banach. Then Y is also Banach.

The proof uses Hahn-Banach.

(e) Every normed space X can be completed to a Banach space.

This fact follows from Hahn-Banach: The natural application $J : X \to X^{**}$ is an injective isometry, and X^{**} is Banach by 2c. Hence, by 2a, the closure of $J(X)$ is the minimal Banach space that "contains" X.

(f) Let X be a vector space of dimension $\aleph_0$ (as a vector space). Then no norm on X can make X complete.

This does not contradict 2d: X still can be included in a Banach space $\widetilde{X}$, but the vector space dimension of $\widetilde{X}$ will be uncountable.

5. **Facts about duality and reflexivity**

(a) A closed subspace of a reflexive Banach space is reflexive.

(b) A Banach space X is reflexive if and only if X^* is reflexive.

(c) If X^* is separable, then X is separable.

(d) **The dual of a subspace and a quotient space:** If M is a closed subspace of X, then M^* is isometrically isomorphic to $X^*/M^\perp$, and $(X/M)^*$ is isometrically isomorphic to $M^\perp$.

Recall that $M^\perp = \{\Lambda \in X^* : \Lambda(x) = 0 \text{ for all } x \in M\}$.