Math 446–646

Important facts about Topological Spaces

- A **topology** on $X$ is a collection $T$ of subsets of $X$ such that:
  1. $\emptyset \in T$
  2. $X \in T$
  3. If $A$ and $B$ are in $T$, then $A \cap B \in T$.
  4. If for each $\alpha \in I$, $A_\alpha \in T$, then $\bigcup_{\alpha \in I} A_\alpha \in T$.
- The elements of $T$ are called **open sets**.
- A subset $F$ of $X$ is closed if $F^c$ is open (i.e., if $F^c \in T$).
- A subset $N \subset X$ is a **neighborhood** of a point $x$ if there is an open set $O \in T$ such that $x \in O \subset N$.
- Let $T_1$ and $T_2$ be two topologies on the same space $X$. If $T_1 \subset T_2$, we say that $T_1$ is coarser than $T_2$, or that $T_2$ is finer than $T_1$.
- Some important examples:
  1. $T_{triv} = \{X, \emptyset\}$, the **trivial topology**.
  2. $T_{dis} = \mathcal{P}(X)$, the **discrete topology**. In the discrete topology, every subset of $X$ is both open and closed.
  3. If $X$ has infinitely many elements, $T_F = \{A \subset X| A^c \text{ is finite or } A^c = X\}$, the **finite-complement topology**.
  4. In $\mathbb{R}$ we consider the **lower-limit topology** $T_l$, whose elements are unions of intervals of the type $(a, b)$, where $a < b$.
  5. If $(X, d)$ is a metric space, the collection $T_d$ of unions of open balls with respect to $d$ is a topology, the **topology induced by the metric** $d$.
    In particular, the topology induced in $\mathbb{R}^n$ by the Euclidean metric is called the **usual topology** on $\mathbb{R}^n$, and denoted by $T_u$.
  6. Let $X$ be a space with an total order relation $\leq$. Assume that $X$ does not have a smallest or a biggest element. Denote by $(a, b) = \{x \in X| a < x \text{ and } x < b\}$. We define the **order topology** $T_<$ on $X$ as follows: A set $O$ is open if it is union of sets of the type $(a, b)$ defined above.
(7) In example (6), assume that $X$ has a smallest element $a_0$ and a biggest element $b_0$. Denote by $[a_0, c) = \{x \in X | a_0 \leq x$ and $x < c\}$ and by $(c, b_0) = \{x \in X | c < x$ and $x \leq b_0\}$. A set $O$ is open if it is union of sets of the types $(c, d), [a_0, c) (c, b_0]$. We also call the topology $T_\prec$, the order topology.

(8) Let $(X, T)$ be a topological space, let $Y \subset X$. Then $Y$ inherits a topology from that of $X$, called $T_Y$, the relative topology in $Y$, as follows: A set $O \subset Y$ belongs to $T_Y$ if and only there exists an open set $U \in T$ such that $O = U \cap Y$.

Note that it is possible for a set $O$ to be open in the topology $T_Y$ and not to be open in the topology $T$.

(9) Let $(X_1, T_1), \ldots, (X_n, T_n)$ be $n$ topological spaces. We define a topology on $\prod_{i=1}^n X_i$, called the product topology $T_{prod}$ as follows: A set $O \in T_{prod}$ if and only if $O$ is a union of sets of the form $O_1 \times \cdots \times O_n$, where $O_i \in T_i$ for each $i = 1, \ldots, n$.

(10) **Identification topologies**: See section 3.8 in the book.

- A collection $B \subset T$ is a basis of the topology $T$ if every open set in $T$ is a union of elements of $B$. A basis is useful to describe a topological space without having to specify all the open sets.
- Bases make it very easy to compare two topologies (instead of comparing all the open sets, we only need check the elements of the basis). Given a basis $B_1$ of the topology $T_1$ and a basis $B_2$ of the topology $T_2$, $T_1 \subset T_2$ if and only if for every element $B \in B_1$ and every point $x \in B$ there exists an element $\tilde{B} \in B_2$ such that $x \in \tilde{B} \subset B$. 

• Let \((X, T)\) be a topological space and let \(A \subset X\). A point \(x \in X\) is in the closure of \(A\) if for every open set \(O\) containing \(x\), \(O \cap A \neq \emptyset\).

• The closure of \(A\) is denoted by \(\overline{A}\). \(\overline{A} = \bigcap F\), where the intersection is taken over all the closed sets \(F\) that contain \(A\) (thus, \(\overline{A}\) is the “smallest” closed set that contains \(A\)). Therefore, \(A\) is closed if and only if \(A = \overline{A}\).

• The interior of \(A\) is the union of all the open sets contained in \(A\). It is denoted by \(A^\circ\). (Thus, \(A^\circ\) is the “biggest” open set contained in \(A\)). \(A\) is open if and only if \(A = A^\circ\).

• The boundary of \(A\) is \(\text{Bdry}(A) = \overline{A} \cap A^c\). A point \(x\) is in the boundary of \(A\) if and only if for every open set \(O\) containing \(x\), \(O \cap A \neq \emptyset\) and \(O \cap A^c \neq \emptyset\).

• Review the properties of interior and closure seen in section 3.6, in the homework assignment for 3.6 and in class.

• A function \(f : (X, T) \to (Y, T')\) is continuous if and only if for every \(O \in T'\), we have \(f^{-1}(O) \in T\) (i.e., the inverse image of any open set is open). Equivalently, \(f\) is continuous if and only if for every closed set \(F\) in \(Y\), \(f^{-1}(F)\) is closed in \(X\).

• Some important continuous functions:
  1. If \(f\) is a constant function, then \(f\) is continuous.
  2. If \(f : (X, T) \to (Y, T')\) is continuous and \(g : (Y, T') \to (Z, T'')\) is continuous, then the composition \(g \circ f : (X, T) \to (Z, T'')\) is continuous.
  3. The identity function \(Id : (X, T) \to (X, T)\) is continuous. However, if we consider two different topologies in \(X\), the identity function \(Id : (X, T) \to (X, T')\) may not be continuous. Example: \(Id : (\mathbb{R}, T_u) \to (\mathbb{R}, T_i)\) is not continuous.
  4. The projection function \(p_k : (\prod_{i=1}^n X_i, T_{\text{prod}}) \to (X_k, T_k)\) is continuous.
  5. Let \((X, T)\) be a topological space. If \(Y \subset X\), then the inclusion function \(i : (Y, T_Y) \to (X, T)\) (defined as \(i(y) = y\)) is continuous.
  6. Let \(f : (X, T) \to (Y, T')\) be continuous. Let \(Z \subset X\). Then the restriction function \(f|_Z : (Z, T_Z) \to (Y, T')\) is continuous.
(7) **The pasting lemma**: Let \((X, T)\) and \((Y, T')\) be topological spaces, and assume that \(X = A \cup B\), where both \(A\) and \(B\) are closed sets. Let \(f : (A, T_A) \to (Y, T')\) and \(g : (B, T_B) \to (Y, T')\) be continuous functions such that, for every \(x \in A \cap B\), \(f(x) = g(x)\). Then the function \(h : (X, T) \to (Y, T')\), defined as

\[
h(x) = \begin{cases} 
  f(x) & \text{if } x \in A \\
  g(x) & \text{if } x \in B
\end{cases}
\]

is continuous.

- A function \(f : (X, T) \to (Y, T')\) is **open** if and only if for every \(O \in T\), we have \(f(O) \in T'\) (i.e., the image of any open set is open).
- An function may be open but not continuous. Also, a function may be continuous but not open.
- A **homeomorphism** is a function \(f : (X, T) \to (Y, T')\) that is bijective, continuous and has a continuous inverse. If there is a homeomorphism between two spaces \((X, T)\) and \((Y, T')\), we say that the two spaces are homeomorphic.
- If \(f\) is bijective, continuous and open, then \(f\) is a homeomorphism.
- We say that a property of a space \((X, T)\) is a **topological property** if every space that is homeomorphic to \((X, T)\) also has that property.
- Topological properties will help us decide if two different topological spaces are homeomorphic or not.
- Some topological properties:
  - (a) A space \((X, T)\) is **metrizable** if there is a metric \(d\) such that \(T_d = T\).
    
    Example: \((X, T_{\text{dis}})\) is always metrizable, with the metric
    
    \[
d(a, b) = \begin{cases} 
  0 & a = b \\
  1 & a \neq b
\end{cases}
\]

- (b) **Property \(T_1\)**: A topological space \((X, T)\) has property \(T_1\) if for any point \(x \in X\), the set \(\{x\}\) is closed.
(c) **Hausdorff property** or **property** $T_2$: A topological space $(X, T)$ has the Hausdorff property if given two different point $x, y \in X$, there exist open sets $O_1, O_2 \in T$ such that $x \in O_1, y \in O_2$ and $O_1 \cap O_2 = \emptyset$.

Any space with the property $T_2$ also has property $T_1$, but the converse is not true.

If $(X, T)$ is metrizable, then it has the Hausdorff property.

Example: In $\mathbb{N}$ we consider the topology $T$ formed by $\emptyset, \mathbb{N}$ and all the sets of the form $\{k, k+1, \ldots\}$. Then $(\mathbb{N}, T)$ is not metrizable because it does not have the Hausdorff property.

(d) The **first-countability axiom**: $(X, T)$ satisfies the first-countability axiom if $X$ has a countable basis for its topology at each one of its points, i.e. at each point $x \in X$ there is a countable collection $\mathcal{B}_x$ of open sets containing $x$, such that for any neighborhood $N$ of $x$, there is an element $B \in \mathcal{B}_x$ such that $x \in B \subset N$.

If $(X, T)$ is metrizable, then it satisfies the first-countability axiom.

(e) The **second-countability axiom**: $(X, T)$ satisfies the second-countability axiom if $X$ has a countable basis for its topology.

A space that satisfies the second-countability axiom also satisfies the first-countability axiom.

Not every metrizable space satisfies the second-countability axiom.

(f) A space $(X, T)$ is **separable** if there is a countable subset $S \subset X$ such that $\overline{S} = X$.

For example, $(\mathbb{R}, T_u)$ is separable, because $\overline{\mathbb{Q}} = \mathbb{R}$. 