Group Actions on Sets

Geometry, Group Theory and Combinatorics

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This talk is broken up into four parts. The first focuses on the definition of a group, examples of groups, and finishes with Lagrange’s Theorem.

The second part of the talk deals with more slightly more advanced group theory. Here we will talk about symmetry groups of convex polygons and a few platonic solids. This portion of the talk will end with the Orbit Stabilizer Theorem.

The third part of the talk has a more combinatorially feel. We dive a bit deeper into the literature and will end with Burnside’s Counting Lemma.
What is a Group?

A group $\Gamma$ is a pair $(S, \ast)$ where $\emptyset \neq S$ (avoid pathologies), $\ast$ is a binary operation on $S$ i.e $\ast : S \times S \rightarrow S$ and the following hold,

- $\exists e \in S$ s.t $\forall s \in S$ we have $e \ast s = s \ast e = s$
- $\forall a \in S, \exists b \in S$ s.t $a \ast b = b \ast a = e$
- $\forall a, b, c \in S$ we have $(a \ast b) \ast c = a \ast (b \ast c)$
Examples of Groups

Level 0: \( G = \{ e \} \)

Level 1: \((\mathbb{Z}, +), (\mathbb{R}, \times, +), (\mathbb{Q}, +), (n\mathbb{Z}, +)\) and \((\mathbb{Z}/n\mathbb{Z}, +)\)

Level 2: \((\text{GL}_n(\mathbb{R}), \cdot_m), (\text{SL}_2(\mathbb{Z}), \cdot_m)\) and \((\text{Perm}\{1, 2, \ldots, n\}, \circ)\)

Level 3: Symmetry Groups, Topological Groups, and Galois Groups
Subgroups

Subgroups are the analogs to subspaces in the sense of vector spaces. In other words a subgroup $H$ of $G$ (denoted $H \leq G$) is a subset of the underlying set for $G$ s.t when equipped with the same binary operation say $\ast$ then $H$ is a group.
Cosets arise from defining equivalence relations on group elements with purpose of taking quotients and retaining information you wish to study. Quotients are the front-runners of topology. For instance, how are $\mathbb{D}^2$ and $S^2$ related? How are $I^2$ and $T^2$ related?
Let $H \leq G$ and consider the product $aH$ where $a \in G$. Sets of the form $aH$ are called left-cosets. We can choose to define an equiv. relation on elements $x, y \in G$ as $x \sim_H y \iff xH = yH$ i.e. $x, y$ generate the same coset. An equivalent definition of this relation is $x \sim_H y \iff xy^{-1} \in H$. 

![Diagram showing cosets of a group $H$ in $G$]
Picture Proof of LT
Lagrange’s Theorem

Theorem: If \( G \) is a finite group and \( H \leq G \) then \( |H| \) divides \( |G| \).

Proof: Let \( G = \{ e, x_1, \ldots, x_{n-1} \} \). Every element \( x_i \in G \) is in some left coset (namely \( x_i \in x_iH \)). However, by a simple counting argument, not every element can generate a distinct left-coset.

Choose \( e, x_1, \ldots, x_k \) (where \( k < n \)) to be the representatives of each coset i.e \( H, x_1H, \ldots, x_kH \). Then \( G = H + x_1H + \cdots + x_kH \).

The map \( \varphi : H \rightarrow \mathcal{L}_H \) by \( \varphi(h) = aH \) is a bijection. Thus we have \( |H| = |aH| \Rightarrow |G| = (k + 1)|H| \) where \( k + 1 = |\mathcal{L}_H| \).
Let $G = \{r_0, r_{90}, r_{180}, r_{270}, m_1, m_2, m_3, m_4\}$ which is the symmetry group of the square. Denote the vertices of the square as 1, 2, 3, 4.
Let $X$ be a set and $G$ be a group. Define $\cdot : G \times X \rightarrow X$ by $g_x = g \cdot x$. If $\forall g \in G, x \in X$ we have $(g_1g_2) \cdot x = g_1 \cdot (g_2 \cdot x)$ and $\exists g_0 \in G$ s.t $\forall x \in X, g_0 \cdot x = x$ then we say $G$ acts on $X$ and will refer to $X$ as a $G$-set.

**Example:** In the case of $G = \text{Square}$ and $X = \{1, 2, 3, 4\}$ we have $G$ acts $X$ wrt symmetries.
In the case of $G = \text{Square}, \ X = \{1, 2, 3, 4\}$, consider the vertex (1) and the continuous action of $r_{90}$ on it. Then the path of (1) is $(1) \mapsto (2) \mapsto (3) \mapsto (4) \mapsto (1)$. This sequence describe elements which we will say are in the orbit of (1). If we denote $\mathcal{O}_x$ as the orbit of $x \in X$ then $\mathcal{O}_x = X$.

Similarly, if we consider the action of $r_0, m_4$ on (1) then $(1) \mapsto (1)$ in each case. We characterize this phenomena as fixing. Thus we can denote the $g \in G$ that fixes an element as $\text{Stab}_x$. In the above example we have that $\text{Stab}_{(1)} = \{r_0, m_4\}$. 

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One can check that $O_x, Stab_x$ are subgroups of the group $G$ for which $X$ is a $G$-set. The more interesting thing here is that in the example above $8 = |G| = |Stab_x||O_x|$.

**Question:** If $G$ is any group and $X$ is a finite $G$-set then is $|O_x| = [G : Stab_x] = |G|/|Stab_x|$?

**Answer:** Yes, and this is called the Orbit-Stabilizer Theorem.
Orbit-Stabilizer Theorem: Let $G$ be a group and $X$ be a finite $G$-set then $|O_x||\text{Stab}_X| = |G|$. 

Proof: The map $g : G/\text{Stab}_x \rightarrow O_x$ defined by $g\text{Stab}_x \mapsto g \cdot x$ is a well-defined, bijection. Thus $[G : \text{Stab}_x] = |O_x|$. Since we have $\text{Stab}_x \leq G$ then $|G| = [G : \text{Stab}_x]|\text{Stab}_x|$ and we are done.
Burnside’s Counting Lemma I

**Motivation**: Suppose you wish to color the vertices of a square either black or white. Then how many distinct coloring are possible?
Let $X_g = \{ x : g \cdot x = x \}$, this is called the fixed point set of $g \in G$. It follows that $X_g \subseteq X$. Our last piece before proving Burnside’s Lemma will be the prove the following statement, let $X$ be a $G$-set and suppose that $x \sim y$. Then $G_x$ is isomorphic to $G_y$. In particular, $|G_x| = |G_y|$. Define the map $\varphi : G_x \to G_y$ by $\varphi(a) = gag^{-1}$ and check that this is in fact an isomorphism.
Burnside’s Counting Lemma

Lemma: Let $G$ be a finite group acting on a set $X$ and let $k$ denote the number of orbits of $X$. Then $k = \frac{1}{|G|} \sum_{g \in G} |X_g|$.

Proof: Consider there number of all $g'$s fixing $x'$s $\in X$. In view of fixed point sets, this number is $\sum_{g} |X_g|$, but in terms of stabilizer subgroups this number is $\sum_{x} |G_x|$. Thus we have equality between these two sums. Moreover, $\sum_{y \in O_x} |G_y| = |O_x||G_x| = |G|$. Hence summing over all $k$ distinct orbits we have $\sum_{g} |X_g| = \sum_{x} |G_x| = k|G|$ and we are done.
Applications

Let $X = \{1, 2, 3, 4, 5\}$ and suppose that $G$ is a permutation group given by $G = \{(1), (13), (13)(25), (25)\}$. Here the orbits of $X$ are \{1, 3\}, \{2, 5\} and \{4\}. The fixed point sets are,

$$X_{(1)} = X, X_{(13)} = \{2, 4, 5\}, X_{(13)(25)} = \{4\}, X_{(25)} = \{1, 3, 4\}$$

We have by Burnside’s Theorem that the number of distinct orbits is $k = \frac{1}{|G|} \sum_g |X_g| = 3$. 

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**Geometric Application**

**Question**: How many distinct colorings (of all vertices) of a square with just the colors black and white?
References


(2) Wikipedia: History of Group Theory