What is a Space Curve?

A space curve is a smooth map \( \gamma : I \subset \mathbb{R} \rightarrow \mathbb{R}^3 \). In our analysis of defining the curvature for space curves we will be able to take the inclusion \((\gamma, 0)\) and have that the curvature of \((\gamma, 0)\) is the same as the curvature of \(\gamma\) where \(\gamma : I \rightarrow \mathbb{R}^2\). Hence, in this part of the talk, \(\gamma\) will always be a space curve.

We will make one more restriction on \(\gamma\), namely that \(\gamma'(t) \neq \vec{0}\) for no \(t\) in its domain. Curves with this property are said to be regular. We now give a complete definition of the curves in which we will study, however after another remark, we will refine that definition as well.

**Definition:** A regular space curve is the image of a smooth map \(\gamma : I \subset \mathbb{R} \rightarrow \mathbb{R}^3\) such that \(\gamma' \neq \vec{0}, \forall t\). Since \(\gamma\) and its image are uniquely related up to reparametrization, one usually just says that a curve is a map i.e making no distinction between the two.

The remark about being unique up to reparametrization is precisely the reason why our next restriction will make sense. It turns out that every regular curve admits a unit speed reparametrization. This property proves to be a very remarkable one especially in the case of proving theorems.

**Note:** In practice it is extremely difficult to find a unit speed reparametrization of a general space curve. Before proceeding we remind ourselves of the definition for reparametrization and comment on why regular curves admit unit speed reparametrizations.

**Definition:** Let \(\gamma : I \rightarrow \mathbb{R}^3\) be a smooth curve. Then \(\tilde{\gamma} : \tilde{I} \rightarrow I\) is said to be a reparametrization of \(\gamma\) if there exists a diffeomorphism \(\Phi : \tilde{I} \rightarrow I\) such that \((\gamma \circ \Phi)(\tilde{t}) = \tilde{\gamma}(\tilde{t})\). This means that \(\gamma\) and \(\tilde{\gamma}\) trace out the same curve.
Recall that if $\gamma$ is a smooth curve, then its arc-length function is given by,

$$s = s(t) = \int_a^t \|\gamma'(u)\| \, du$$

Therefore, $s : I = [a, b] \to s([a, b]) = \tilde{I}$. Since $\gamma$ is regular and “$f$” is monotonic, it follows that $s$ is invertible. Even more, by the inverse function theorem, since $s'(t) = \|\gamma'(t)\| \neq 0$, $s$ is a local diffeomorphism.

It follows that $(\tilde{\gamma} \circ s)(t) = \gamma(t)$ or simply $\tilde{\gamma}(s) = \gamma(t)$, keeping in mind that $s$ is a function of $t$. Observe that $\gamma(s) = (\gamma \circ s^{-1})(\tilde{t}) = \gamma(t)$ i.e $\gamma \circ s$ is a reparametrization of $\gamma(t)$ with $\Phi = s^{-1}$. By the chain rule we have,

$$\gamma'(s) = (\gamma \circ s^{-1})'(\tilde{t}) = \gamma'(s^{-1}(\tilde{t})) \cdot (s^{-1})'(\tilde{t})$$

$$= \gamma'(t) \cdot \frac{1}{s'(s^{-1}(\tilde{t}))}$$

$$= \gamma'(t) \cdot \frac{1}{s'(t)}$$

$$= \gamma'(t) \cdot \frac{1}{\|\gamma'(t)\|}$$

Hence, $\|\gamma'(s)\| = 1$ i.e it is a unit speed reparametrization of $\gamma(t)$. Observe that regularity was crucial to the argument since $\|\gamma'(t)\|$ must have nonzero norm in order for $\gamma'(s)$ to be defined. Henceforth, we will assume $\gamma$ is parametrized by the arc-length parameter $s$ i.e we will write $\gamma(s)$ instead of $\gamma(t)$ or if the context is clear, we simply say $\gamma$ is unit speed and make no fuss about the variable.
Curvature of Space Curves

We are now ready to define the curvature for a unit speed space curve. Our definition will involve a function $\kappa$ parametrized by a unit speed parameter $s$, whereas a general regular curve it may not already be parametrized by arc-length.

**Comment:** There is a way to derive $\kappa(t)$ without adhering to a unit speed reparametrization, however the interest in that case is purely for computational reasons. We will give the formula below, but that is all we will do. The argument is given in almost any multivariable calculus text.

**Definition:** Let $\gamma : I \to \mathbb{R}^3$ be a unit speed curve. Then we define the curvature of $\gamma(s)$ to be $\kappa(s) = \|\gamma''(s)\|$.

This definition adheres to our geometric intuition of what it means for a curve to bend, namely that there is a change in the tangent vector. From a physics view we know that it is acceleration which causes a particle’s velocity to change direction.

**Physics Viewpoint:** Hence, in order to define the “bentness” of a curve, we’ve associated to it the force required (acceleration) to take a particle traveling in the direction of $\gamma'(s)$ off-track.

Observe that if $\gamma(t)$ is not unit speed, or doesn’t have a constant velocity, then $\kappa(t)$ is not well defined. Mathematically this can be shown by using the chain rule, however we conceptualize this fact by observing that, if $\gamma(t)$ is traveling very fast along $\gamma$, then its velocity will be changing more quickly than if it were traveling very slowly. We now give the formula for $\kappa(t)$ where $t$ is not a unit-speed parameter.

**Theorem:** Let $\gamma : I \to \mathbb{R}^3$ be a smooth regular curve. Then the curvature at $\gamma(t)$ is given by,

$$
\kappa(t) = \frac{\|\gamma'(t) \times \gamma''(t)\|}{\|\gamma'(t)\|^3}
$$
Frame Fields

Let $\gamma$ be a unit speed curve. Then $\gamma'$ induces a velocity vector field along $\gamma$. Since $\gamma \cdot \gamma = 1$, we have $\gamma \cdot \gamma' = 0$ i.e the acceleration and velocity vectors are perpendicular where $\gamma''$ points in the direction in which $\gamma$ is turning.

Here we have $\{\gamma', \gamma''\}$ induce a frame at $\gamma(t)$ in the sense that any vector $v \in T_{\gamma(t)}\gamma$ we have $v = a'\gamma' + b'\gamma''$ where $a', b' \in \mathbb{R}$. This is true due to the fact that,

$$v = a'\gamma' + b'\gamma'' \Rightarrow v \cdot \gamma' = a' \text{ and } v \cdot \gamma'' = b'$$

Depending on the orientation of $\gamma$ the frame $F = \{\gamma', \gamma''\}$ is either positively or negatively oriented. If we orient $\gamma$ with the notion that increasing is from left to right, then $F$ induces the positive orientation and vice-versa.

By the remark above, $F' = \{e^1 = \gamma', e^2 = \gamma'', e^3 = \gamma \times \gamma''\}$ becomes a frame for $\gamma$ where now we allow $v \in \mathbb{R}^3$ i.e not confined to the tangent plane at $\gamma(t)$. Again with the orientation of this frame depending the orientation given to $\gamma$. Hence, given any $v$ with tail at $\gamma(t)$ we have by orthonormal expansion,

$$v = \sum_j \omega_j e^j \quad \text{where} \quad \omega_j = v \cdot e^j$$

We will see soon enough that studying curves by moving frame fields fitted to the curve is extremely fruitful and leads us to a classification. In what comes to follow we will set $T = \gamma'$, $N = \gamma''/\kappa$ and $B = T \times N$. The reason for introducing what seems to be “unnecessary” notation is because although $\gamma(t)$ is just a regular curve, we will adhere to the fact that it admits a unit speed parametrization and prove theorems about $\gamma(s)$. 


The Frenet-Serret Formulas

Let \( \gamma : I \to \mathbb{R}^3 \) be a unit speed curve. We now wish to compute \( T', N' \) and \( B' \). The idea is to compute these quantities with respect to the frame given by \( T, N, B \) at \( \gamma(t) \). This is precisely what we want since everything is fitted to the curve.

Now that we’ve convinced ourselves that this is what we must do, the computation follows from orthonormal expansion. Below, we will define \( (N \cdot B') = \tau \) and will wait to compute this quantity.

\[
T' = (T' \cdot T) T + (T' \cdot N) N + (T' \cdot B) B = \kappa N
\]

\[
N' = (N' \cdot T) T + (N' \cdot N) N + (N' \cdot B) B = -\kappa T + \tau B
\]

\[
B' = (B' \cdot T) T + (B' \cdot N) N + (B' \cdot B) B = -\tau N
\]

Again we remark that the beauty in these formulas is not apparent for practical purposes, but in reference to general questions about curves, they will tell you almost everything. Before we embark on the journey of classifying curves, we given the promised formula for \( \tau \) which is the torsion.

**Theorem:** Let \( \gamma \) be a regular curve (not necessarily unit speed) then the torsion \( \tau \) is given by the formula below.

\[
\tau = \frac{(\gamma' \times \gamma'') \cdot \gamma'''}{\|\gamma' \times \gamma''\|^2}
\]

The formula above is for computational purposes and tell us nothing geometrically. However, from our original definition \( \tau = N \cdot B' \) i.e \( \tau \) measures the amount that \( B \) rotates in the direction of \( N \) and this is the interpretation we hold on to.
Classifying Planar and Space Curves

It wasn’t by coincidence that we defined $\kappa$ and $\tau$ the way we did. We’ve given geometric reasons for $\kappa$’s definition, but why $\tau$’s. In the case for a planar curve, it is completely characterized by how it curves at each point. Now if we allow $\gamma$ to be a space curve, it no longer has to remain in a single plane and $\tau$ measures how $\gamma$ bends away from its normal plane.

We can now “classify” all planar and space curves. This will be done in a very “hand wavy” sort of fashion, but strong intuition in mathematics means everything. We present the cases that we’ll be analyzing for our classification below.

**Classification of Planar Curves:** The geometric makeup of a planar curve is determined by $\kappa$ so what does $\gamma$ look like geometrically, if we let $\kappa \equiv C, C \in \mathbb{R}$? $\kappa > 0$? $\kappa < 0$? $\kappa \equiv 0$?

**Classification of Space Curves:** The geometric makeup of a planar curve is determined by both $\tau, \kappa$. Hence, we combine the cases above with $\tau \equiv 0, \tau \equiv K, \tau > 0, \tau < 0$. 
The Shape Operator

Let $M \subset \mathbb{R}^3$ be a surface. Suppose we can orient a neighborhood $V$ of $p \in M$ with the outward pointing normal vector field $U$ i.e $U : M \to T_p M$. Hence the quantity $U(p + tv)'(0)$ measures the initial rate of change for $U$ in the direction of some unit vector $v$.

Note that a change in direction of the unit normal implies a change in placement of the tangent plane. Therefore, $U(p + tv)'(0)$ describes how the tangent planes along $M$ change in the direction of $v$ i.e describing the shape of $M$ in this direction. Thus we define $S_p(v) = \nabla_v U = U(p + tv)'(0)$ to be the shape operator.

Let $U$ and $V$ be defined as above and suppose now that $\alpha : (-\epsilon, \epsilon) \to V$ is a smooth curve with $\alpha(0) = p$. Then we have:

$$\alpha' \cdot U = 0 \Rightarrow \alpha'' \cdot -U = \alpha' \cdot U' = \alpha' \cdot S(\alpha')$$

where we let $U' = \nabla_v U$. Since $\alpha''$ measures the bend of $\alpha$ away from $\alpha' \in T_p M$, then $\alpha'' \cdot -U$ denotes how much $\alpha$ bends away from the tangent plane.

Choosing $\alpha$ to be a unit speed curve, we consider the Frenet apparatus at $p$ and notice that:

$$\alpha'' \cdot -U = \kappa(0)N(0) \cdot -U = \kappa(0) \cos \nu = \pm \kappa \sigma(0) \cos \nu$$

since for unit speed curves $|\alpha''| = \kappa$ and a priori $\alpha'' = \kappa \sigma n \sigma$ i.e $\kappa = \pm \kappa_s$. In the above $\nu$ represents the angle between $N$ and $-U(p)$.

We can actually arrange it so that $\nu = 0$ or $\pi$. Let $u$ be a unit vector and $P$ be the plane containing the lines $U(p)t + p$ and $p + tu$. Let $\alpha$ be the same smooth curve as above, but take $\alpha'(0) = u$. Observe that since plane $P$ carves out a section in $M$ i.e $\alpha$ is bending in $P$. This implies $\alpha''(0) \in P$ and so $\alpha''(0) = \pm U(p)$. Hence we have $S_p(\alpha') \cdot \alpha' = \pm \kappa \sigma$. Our last remark is that the shape operator is symmetric i.e $S(v) \cdot u = S(u) \cdot v$. 

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Normal Curvature

The section in which the plane $P$ determined by $U(p), u$ carve out in $M$ is called a normal section. We let $k(u) = S(u) \cdot u$ denote the curvature at $p$ in the direction of $u$. We call $k(u)$ the normal curvature at $p$. We also remark that $k(u) = k(-u)$.

Note that $k : S^1 \to \mathbb{R}$ and so it realizes a min and max. The minimum and maximum values $k_1, k_2$ are called the principle curvatures. The directions in which they occur are called principle directions and the vectors with these directions are called principle vectors.

Before giving a few examples, we expand a bit on a geometric interpretation of the sign of $k(u)$. If $k(u) > 0$ then $N(0) = U(p)$ which implies the curve $\sigma$ is bending upward away from $T_pM$. If $k(u) < 0$ then $N(0) = -U(p)$ which implies $\sigma$ is bending downward away from $T_pM$.

Example: The unit sphere given by $x^2 + y^2 + z^2 = 1$ has $k_1 = k_2 = 1$. This follows from the fact that each normal section is a unit circle.

Example: Note that every curve which is described by a normal section of direction $u \neq \lambda e^2$ bends away from its tangent plane i.e $\kappa(u) = \kappa_\sigma(0) < 0$. For those sections in the direction of $u = \lambda e^2$ we have $\kappa_\sigma(0) = 0$ since they are lines i.e $k_2 = 0$. 
EigenValues and EigenVectors of $S_p(u)$

We will now analyze the eigenvectors, eigenvalues of the shape operator. It turns out that the eigenvalues are precisely the principle curvatures and the eigenvectors are the corresponding principle directions.

**Definition:** A point $p \in M \subset \mathbb{R}^3$ is umbilic provided the normal curvature $k(u)$ is constant on all unit tangent vectors $u$ at $p$.

Suppose $k$ takes on it maximum value at $k_1$ i.e $k_1 = k(e_1) = S(e_1) \cdot e_1$. Let $e_2$ denote a unit tangent vector orthogonal to $e_1$ at $p$ as well. Then given any unit vector $u$ at $p$ we have $u = u(\nu) = ce_1 + se_2$ where $c = \cos \nu$ and $s = \sin \nu$.

Thus to each normal section carved out in the direction of $u$, we can associate that to the direction of the angle $\nu$ make in the plane spanned by $e_1(p)$ and $e_2(p)$. Hence, the normal curvature becomes a function on the real line i.e $\kappa(\nu) = \kappa(u(\nu))$. For $1 \leq i, j \leq 2$ we define $S_{ij} = S(e_i) \cdot e_j$. It follows that $k_1 = S_{11}$ and by symmetry of the operator, $S_{12} = S_{21}$.

$$
\kappa(\nu) = S(ce_1 + se_2) \cdot (ce_1 + se_2) = c^2 S_{11} + 2scS_{12} + S^2 S_{22}
$$

$$
\frac{dk}{d\nu}(\nu) = 2sc(S_{22} - S_{11}) + 2(c^2 - S^2)S_{12}
$$

We first observe that $k(\nu)$ has a maximum at $\nu = 0$ since this implies $c = 1, s = 0 \Rightarrow u(0) = e_1$. It now follows that:

$$(dk/d\nu)(0) = 0 \Rightarrow S_{12} = S_{21} = 0
$$

If $p$ is umbilic then by orthonormal expansion we have $S(e_1) = S_{11}e_1$ and $S(e_2) = S_{22}e_2$ i.e $S = kI$ where $I$ denotes the identity matrix. If $p$ is no umbilic then we have $k(\nu) = c^2 k_1 + s^2 S_{22}$. Since the max occurs a $k_1$ and $k(\nu)$ is not constant, $k_1 > S_{22}$. Hence $(dk/d\nu)(\nu) = 0 \iff c = 0, s = \pm 1$. That is, we recover the min in the direction of $e_2$. The matrix $S = (S_{ij})$ therefore is given by:

$$
S = \begin{pmatrix}
    k_1(p) & 0 \\
    0 & k_2(p)
\end{pmatrix}
$$
Gaussian and Mean Curvatures

The Gaussian curvature of \( M \subset \mathbb{R}^3 \) is given by \( K = \det(S) \). The mean curvature of \( M \) is \( H = 1/2 \text{trace}(S) \). By the above we have that:

\[
K(p) = \det(S(p)) = \det\begin{pmatrix} k_1(p) & 0 \\ 0 & k_2(p) \end{pmatrix} \Rightarrow K = k_1k_2 \text{ & } H = \frac{1}{2}(k_1 + k_2)
\]

**Summary**: We end our discussion by giving geometric interpretation to the sign of the Gaussian Curvature.

If \( K(p) > 0 \) then the principle curvatures have the same sign. Thus either \( k(u) > 0 \) or \( k(u) < 0 \) for all unit vectors \( u \). This follows from the fact that \( u(\nu) = k_1 \cos^2 \nu + k_2 \sin^2 \nu \). Thus \( M \) is bending away from its tangent plane.

![Image](image1.png)

If \( K(p) < 0 \) then the principle curvatures have opposite signs i.e \( M \) is bending in opposite ways in its principle directions, creating a saddle-like obstruction near \( p \).

![Image](image2.png)

If \( K(p) = 0 \) then we have two cases. If \( k_1(p) \neq 0, k_2(p) = 0 \) then we get a cylindrical approximation near \( p \). If \( k_1(p) = k_2(p) = 0 \) then \( M \) is a plane near \( p \) and we get no information.

![Image](image3.png)