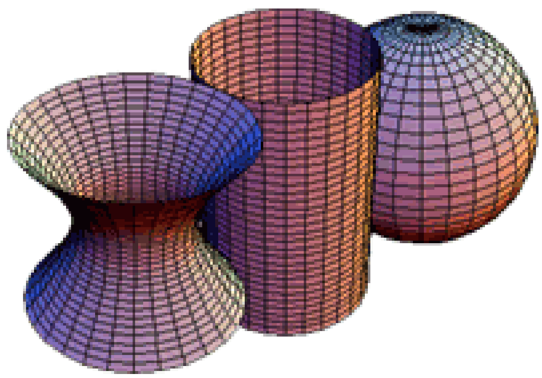
Surfaces, Orientation and Integration

North Dakota State

Faraad Armwood



Abstract: At first glance these three topics seem to have no relation, however we will show that if you wish to integrate over a surface Σ then to get a well defined definition, we must have that Σ is orientable. The question now becomes, what does it mean for a surface to be orientable? Moreover, what is a surface?

Informal Discussion on Manifolds

Since surfaces are the special case of more general objects called manifolds, we will first begin with an informal discussion on manifolds. The reason for the informality here is because we would like to build geometric intuition for what a surface is before we need to rigorously define it to transition towards the topics of orientation and integration. An n-dimensional manifold M is an object which locally resembles n-dimensional euclidean space. For short we say M is an n-manifold.

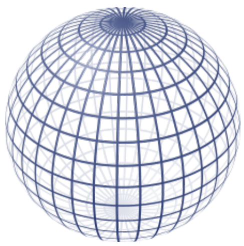
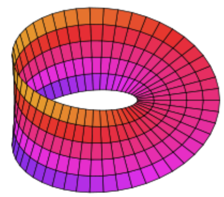
Consider the case when $n = 3$. Can you give an example of a 3-manifold? If you're struggling with this example, look around and ask yourself, how many dimensions can I see? The answer is three. You can see the flat surface below you, which only has two dimensions, but you can also see above the surface and this is your third dimension.

Ideally, it is extremely hard to study 3-manifolds and it has been suggested that the best way to do so is to in-vision yourself inside the manifold (as we naturally do with our planet). The reason one would take this approach of looking from the inside out is because we can only process three-dimensions i.e there is no clear way to view a manifold globally from the outside.

Although there are ways to study three-manifolds, these methods prove to be extremely technical, however the mathematics is beautiful. As three-dimensional beings, we have a much easier time observing objects which are locally two-dimensional. Thus, so we opt to take the case when $n = 2$ and now we begin.

Surfaces

A surface Σ is space such that at each point $p \in \Sigma$, there exists a neighborhood U of p which resembles $\mathbb{R}^2$. Some examples of surfaces are given below.


Now that we have some geometric intuition for what a surface is, the question now becomes, how does one actually show that a space defines a surface? For this we need to transition to a more rigorous definition of what it means to be a surface. What we can extrapolate from our informal definition is that Σ is assembled by patches. It turns out that if we can correctly define what it means for the surface to be covered by patches, we are 1/3 away from being done with our definition.

For Σ to be covered by patches which resemble $\mathbb{R}^2$, then that means we have maps $\phi_j : U_j \to \phi_j(U_j) \subset \Sigma$. It is natural to want U_j and its image $\phi_j(U_j)$ to be open sets since ultimately we would like to do calculus on surfaces. One way to ensure that $\phi_j(U_j)$ is an open set is to require that ϕ_j be a homeomorphism. Informally, a homeomorphism will say that $\phi_j(U_j)$ is open and $\phi_j^{-1}(\phi_j(U_j)) = U_j$ i.e we have an honest to goodness cover of Σ by patches. In particular, the covering is done in an 1-1 fashion i.e we don't have patches piling up on each other. Let us now document our progress on defining a surface;

1) There exist a collection $\{\phi_j : U_j \to \Sigma \mid \phi_j$ **is a homeomorphism**$\}$

2) $\Sigma \subset \bigcup_j \phi_j(U_j)$

3) ??

The last requirement is the most important. It is a statement about how we wish charts ϕ_j, ϕ_i which intersection along their domains to relate to each other. This relationship is actually one that we impose due to an intrinsic property that we wish to maintain, namely the velocity vector emanating from a point, should be independent of its coordinate representation. For those familiar with the terminology, this means that we would want the tangent space a $\phi_i(p)$ which is given in some coordinates, to be the same as the tangent space at $\phi_j(p)$ which is given in another set of coordinates. We will make this more precise in the following next section.

The Derivative as a Linear Map

Let $f : U \subset \mathbb{R}^m \to \mathbb{R}^n$ be a smooth map and $\gamma : (-\epsilon, \epsilon) \subset \mathbb{R} \to U$ be a smooth curve, $p = \gamma(0)$. Then we define the differential of f at p to be the linear map $d_pf(w) = (f \circ \gamma)'(0)$, where $w = \gamma'(0)$. We will now show that d_pf is independent of the curve γ passing through p and d_pf is actually a linear map.

Proof: For simplicity we let $m = 2, n = 3$. Let u, v be the coordinates in $\mathbb{R}^2$ and x, y, z be coordinates in $\mathbb{R}^3$. If we let $\alpha : (-\epsilon, \epsilon) \to U$ be any smooth map, then we have that$\alpha(t) = (u(t), v(t))$. Let $e^1 = (1, 0), e^2 = (0, 1)$ and $f^1 = (1, 0, 0), f^2 = (0, 1, 0), f^3 = (0, 0, 1)$ be the basis vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$ respectively. Thus, $\alpha'(0) = u'(0)e^1 + v'(0)e^2$ and we have;

$$(f \circ \alpha)(t) = (x(u(t), v(t)), y(u(t), v(t)), z(u(t), v(t)))$$

Now by the chain rule and evaluating at $t = 0$, we have;

$$(f\circ\alpha)'(0) = \left(\frac{\partial x}{\partial u}\frac{du}{dt} + \frac{\partial x}{\partial v}\frac{dv}{dt}\right) f^1 + \left(\frac{\partial y}{\partial u}\frac{du}{dt} + \frac{\partial y}{\partial v}\frac{dv}{dt}\right) f^2 + \left(\frac{\partial z}{\partial u}\frac{du}{dt} + \frac{\partial z}{\partial v}\frac{dv}{dt}\right) f^3$$

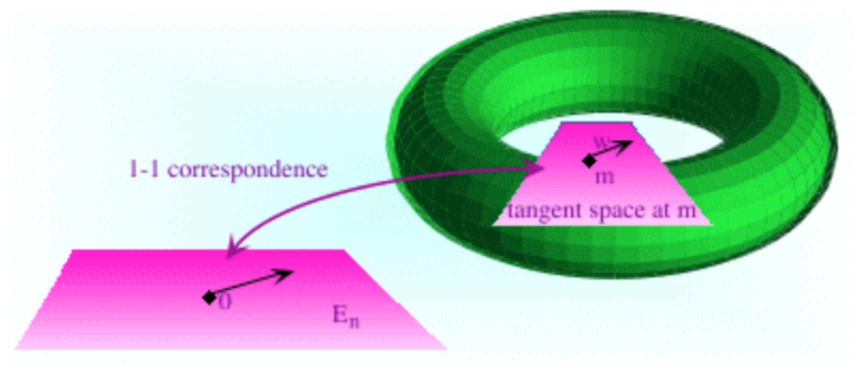
$$= \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix} \begin{pmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \end{pmatrix} = d_pf(w)$$

The connection between the last two sections should be becoming a bit more clear. The underlying condition that we are after is that if the transition map $\phi_{ij} = \phi_i \circ \phi_j^{-1}|_{\phi_j(U_{ij})}$ is a diffeomorphism then the tangent spaces are the same.

Tangent Space for Surfaces

We define the tangent space $T_q\Sigma = \{v : v \text{ is a vector at } q\}$ where $q \in \Sigma$. Let $f : U \subset \mathbb{R}^2 \to \Sigma \subset \mathbb{R}^3$ be a smooth map and U and open subset. Let $p \in U$ then d_pf is a linear map on tangent vectors i.e $d_pf : T_p\mathbb{R}^2 \to T_q\Sigma$ where we let $f(p) = q$. We now have another definition of the tangent space, namely:

$$T_q\Sigma = \mathbf{im}(d_pf)$$



Recall that given a $w = du/dt\ e^1 + dv/dt\ e^2$ we have;

$$d_pfw = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix} \begin{pmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \end{pmatrix}$$

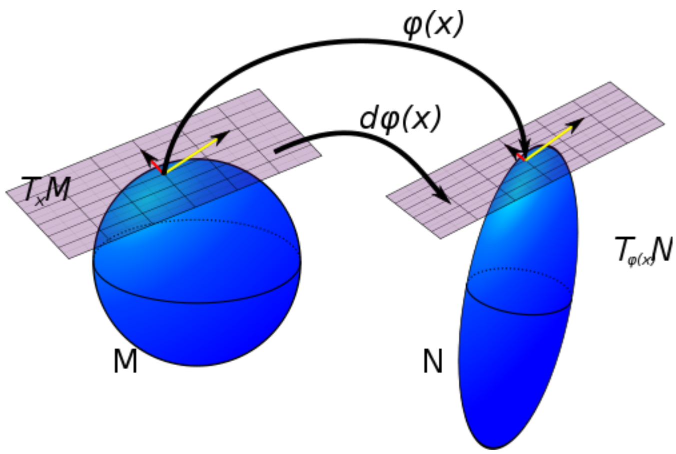
Letting $d_pfe^1 = \partial f/\partial u$ and $d_pfe^2 = \partial f/\partial v$ we have that any vector in the image of d_pf is of the form;

$$\lambda_1 \frac{\partial f}{\partial u}\Big|_p + \lambda_2 \frac{\partial f}{\partial v}\Big|_p \Rightarrow T_q\Sigma = \textbf{span}\left\{ \frac{\partial f}{\partial u}\Big|_p, \frac{\partial f}{\partial v}\Big|_p \right\}$$

i.e $T_q\Sigma$ is a 2-dimensional vector space. Hence, when we said the tangent spaces at points lying in a shared domain of two charts should be the same, we mean up to a linear isomorphism. We are now ready for the claim.

Fact: If $f, g : U \cap V \to \Sigma$ are homeomorphisms and $f \circ g^{-1}$ is a diffeomorphism then $d_{g(p)}(f \circ g^{-1}) : T_{g(p)}\Sigma \to T_{f(p)}\Sigma$ is an isomorphism.

Proof: First we must strengthen the condition of f, g, making them $\mathcal{C}^1$ diffeomorphisms. From this we have that $d_{g(p)}g^{-1} = (d_p g)^{-1}$. Now applying the chain rule we have $d_{g(p)}(f \circ g^{-1}) = d_p f \circ d_{g(p)}g^{-1} = d_p f \circ (d_p g)^{-1}$. The rest is left to the reader.



Informal Discussion on Orientation

To begin talking about orientation of surfaces, we will invite ourselves to an informal discussion about orientation in the plane. First, let us begin with the literal definition of the word. Orientation is choice which helps one determine her/his relative position in comparison to other observes objects.

Consider two 2-dimensional beings in the plane. Observe that to each other, everything appears to be a vertical line. Hence, one can really see how orientation is important since 2-dimensional beings can't see their world from the outside-in, a world of confusion will surmount if a choice of orientation is made, but changes after some time during a morning stroll.

We will now mathematically quantify the possible choices in the plane. Let p be a point in the plane. If p chooses to let the direction vector $(-1, 0)$ as her/his forward = positive orientation, we will call this the counterclockwise direction. The reason to the same is simply because any particle rotating in the direction of $(-1, 0)$, by choice of p, is moving forward. Similarly, we let the opposite direction be the clockwise direction = negative orientation. We remark that there is no reason why one cannot let the clockwise direction be the positive orientation because again, it is all a matter of choice.

Observe that now when we transition to defining an orientation in 3-space, we face the same dellima as our 2-dimensional beings. Our choice here now become, what is outside and what is inside? To a 3-dimensional being it seems pointless to need to decide such a thing since it is clear. However, this is exactly the same situation we encountered with the lower dimensional example. Although it is clear to us how to deflame the dispute between two 2-dimensional beings who have chosen opposite orientations, it take much more thought for 3-dimensional beings to explain why an observable is the outside or inside of something. Although, a bit hard to understand, we see that outside and inside is a choice and now we will make this precise.

On your right hand (with palm toward eyes), let your index finger denote the y-axis and middle finger denote the z-axis. We say that one has chosen the position orientation in 3-space, if when you curl the x-axis finger to the y-axis finger, your thumb points in the outward direction. Mirroring this process with the left hand, the thumb will point in its outward direction. The difference between the two choices is that the right hand's outward is the left's inward and vice-versa. We call the first orientation (pos.) the clockwise orientation and the latter, the counterclockwise orientation.

Orientation of Surfaces

In every discussion which follows, we will choose the positive orientation. Let (ψ, U) be a chart of a surface Σ i.e $\psi : U \to \Sigma$ is a $\mathcal{C}^1$ diffeomorphism. From our above remarks we have that;

$$T_{\psi(p)}\Sigma = \textbf{span}\left\{ \frac{\partial f}{\partial u}\bigg|_p, \frac{\partial f}{\partial v}\bigg|_p \right\}$$

Observe that for a patch on Σ, there are a ton of vectors which emanate from $\psi(p)$ which do not lie in $T_{\psi(p)}\Sigma$, but we need to only choose one. Let us define the outward-pointing normal vector at $\psi(p)$ to be;

$$\mathbf{n}_p := \frac{\partial f}{\partial u}\bigg|_p \times \frac{\partial f}{\partial v}\bigg|_p$$

It follows that $\mathbf{n}_p \notin T_{\psi(p)}$ since it is perpendicular to all other vector also emanating from $\psi(p)$. Hence, we will say that a surface Σ is orientable if $\mathbf{n}_p$ is always well-defined. In this definition, we allow any other vector at $\psi(p)$ of the form $X = \beta n_p$ (where $\beta \in \mathbb{R}_{>0}$) to determine the orientation at $\psi(p)$. Notice that for this definition we need $\mathbf{n}_p$ to always be non-zero. We add this requirement to our definition of a surface.

Remark: Requiring $\mathbf{n}_p \neq 0$ is not the same as it being well-defined.

Remark: Observe that $\mathbf{n}_p \in T_{\psi(p)}\mathbb{R}^3$. For those familiar with the language of differential forms, what we've just stumbled upon is the equivalent statement of orientability as presented in differential topology, namely: M^n is said to be orientable if there exists a non-vanishing n-form. Here the non-vanishing 2-form will be "$\mathbf{n}_p$".

Integration on Surfaces

We begin with a sketch of the change of variable formula. Suppose the map $G : (a, b) \times (c, d) = D \subset \mathbb{R}^2 \to G(D) = D_0 \subset \mathbb{R}^2$ is a $\mathcal{C}^1$ diffeomorphism. Let $f : D_0 \to \mathbb{R}$ be a smooth map. Suppose we take D to be a sufficiently small rectangle, then we have;

$$\iint_{D_0} f\ dxdy \approx f(p) \cdot \mathbf{Area}(D_0)$$

$$= f(G(p_0))|\det(\mathbf{Jac}(G_{p_0}))| \cdot \mathbf{Area}(D)$$

$$= \iint_D (f \circ G)|\det(\mathbf{Jac}(G))|\ dudv$$

To obtain the result for a general open subset U of $\mathbb{R}^2$, we partition U into small rectangles (patches) then we have;

$$\iint_{D_0} f\ dxdy \approx \sum_{i,j} f(p_{ij}) \cdot \mathbf{Area}(R_{ij})$$

$$= \sum_{i,j} f(G(\tilde{p})_{ij}))|\det(\mathbf{Jac}(G_{\tilde{p_{ij}}})| \cdot \mathbf{Area}(\tilde{R}_{ij})$$

$$= \iint_D (f \circ G)|\det(\mathbf{Jac}(G))|\ dudv$$

Let G be the map defined above with (x, y) being the coordinates on D and (u, v) being the coordinates on $G(D) = D_0$. Then we have:

$$\mathbf{Jac}(G) = \begin{pmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{pmatrix}$$

Linear and Orientation Preserving Maps

For simplicity, let G be a linear map from D to D_0, that is a map s.t $G(\alpha\mathbf{x}+\mathbf{y}) = \alpha G(\mathbf{x}) + G(\mathbf{y})$, for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$ and $\alpha \in \mathbb{R}$. We can take $G(x, y) = (Ax + By, Cx + Dy) = (u, v)$ where $A, B, C, D \in \mathbb{R}$.

Suppose again that D, D_0 are sufficiently small rectangles in the plane. Suppose also that G is not orientation preserving. What this means is that if we let D be given the counterclockwise orientation $[e^1, e^2]$, then D_0 has the clockwise orientation $[-e^1, e^2]$. Using our new notation, we can denote this as: $[e^1, e^2] \mapsto [-e^1, e^2]$. Hence, if we let $(u, v) \in (\mathbb{R}^2, [e^1, e^2])$ then we have;

$$(u, v) = ue^1 + ve^2 \longmapsto -ue^1 + ve^2 = (-u, v)$$

Thus, if we let $R = [a, b] \times [c, d]$ be the unit square in $(\mathbb{R}^2, [e^1, e^2])$ lying in the first quadrant then;

$$\mathbf{Area}(G(R)) = |\det(\mathbf{Jac}(G))| = \left|\begin{pmatrix} -a & b \\ -c & d \end{pmatrix}\right| = -\det(\mathbf{Jac}(G))$$

The connection here is that, for us G will be our transition map ϕ_{ij} and what the above tells us it that;

$$\iint_{D_0} f \; dxdy = \pm \iint_{D} (f \circ \phi_{ij}) \; \det(\mathbf{Jac}(\phi_{ij})) \; dudv$$

depending on whether ϕ_{ij} is orientation preserving or not. What we've show above (well at least for linear maps) is that orientation reversing maps have negative determinant. Hence if we want to get a well-defined definition for surfaces, we require that $\mathbf{Jac}(\phi_{ij}) > 0$. However since approximate the patches on a surfaces by $\|\mathbf{n}_p\|$, this statement is equivalent to saying that we can orient our surface with a choice of an outward pointing normal vector i.e Σ is orientable.