Interior Multiplication

Let $\omega \in \Omega^k(M)$ and $v_2, ..., v_k \in TM$. We define the interior multiplication or contraction along a tangent vector $X$ to be:

$$\iota_X \omega(v_2, ..., v_k) = \omega(X, v_2, ..., v_k) \in \Omega^{k-1}(M)$$

If we transition back to when differential forms we just co-vectors then one can show that for 1-covectors $\alpha^1, ..., \alpha^k$ on a vector space $V$ and $v \in V$:

$$\iota_v (\alpha^1 \wedge \cdots \wedge \alpha^k) = \sum_{i=1}^{k} (-1)^{i-1} \alpha^i(v) \alpha^1 \wedge \cdots \wedge \tilde{\alpha}^i \wedge \cdots \wedge \alpha^k$$

where $\tilde{\alpha}^i$ means that we delete this term in the sum. We also have a few more properties:

- $\iota_X \circ \iota_X = 0$
- $\omega \in \Omega^k(M)$ and $\tau \in \Omega^l(M)$ then $\iota_X (\omega \wedge \tau) = (\iota_X \omega) \wedge \tau + (-1)^k \omega \wedge \iota_X \tau$

Using interior multiplication, we can show that $S^n$ is an orientable manifold. Let $X = \sum_i x_i \partial^i$ be the radial vector field. Then $X|_{S^n}$ becomes a normal vector field on $S^n$. Let $\omega = \iota_X (dx^1 \wedge \cdots \wedge dx^{n+1})$ and $v_2, ..., v_n$ be a basis for $T_p S^2$. Then $\{X_p, v_2, ..., v_n\}$ is a basis for $T_p \mathbb{R}^{n+1}$ and hence:

$$\omega(v_2, ..., v_n) = \iota_X (dx^1 \wedge \cdots \wedge dx^{n+1})(v_2, ..., v_n)$$

$$= dx^1 \wedge \cdots \wedge dx^{n+1}(X, v_2, ..., v_n)$$

$$\neq 0$$
Non-vanishing 2-form on $S^2$

Although the Hairy Ball Theorem holds for all even-dimensional spheres, we will restrict our attention to the 2-sphere. A non-vanishing closed 2-form on $S^2$ is given by the contraction along the radial vector $X$ i.e:

$$\omega = \iota_X(dx^1 \wedge dx^2 \wedge dx^3) = x_1 dx^2 \wedge dx^3 - x_2 dx^1 \wedge dx^3 + x_3 dx^1 \wedge dx^2$$

by abuse of notation we let $x^i = x^i|_{S^2}$.

Using the parametrization $F : D = (0, 2\pi] \times [0, \pi] \rightarrow \mathbb{R}^3$ defined by:

$$(\theta, \phi) \mapsto \begin{pmatrix}
\cos \theta \sin \phi \\
\sin \theta \sin \phi \\
\cos \phi
\end{pmatrix} = \begin{pmatrix}
x \\
y \\
z
\end{pmatrix}$$

one can easily show that:

$$\int_{S^2} \omega = \int_D F^* \omega \neq 0$$
Stoke’s Theorem and Homotopic Invariance

Before coming to the final result, we need to establish a few lemmas. First and foremost be recall **Stoke’s Theorem**: if \( M \) is an oriented, compact \( n \)-manifold with boundary and \( \omega \in \Omega^{n-1}(M) \) then:

\[
\int_{\partial M} \omega = \int_M d\omega
\]

**Lemma 1**: Let \( F : \partial W \to Y \) be a smooth map that extends to all of \( W \) and \( \omega \) is a closed \( k \)-form on \( Y \) with \( k = \dim \partial W \) then;

\[
\int_{\partial W} F^* \omega = 0
\]

Proof: The result follows from Stoke’s Theorem.

\[
\int_{\partial W} F^* \omega = \int_{\partial W} d(F^* \omega) = \int_{\partial W} F^*(d\omega) = \int_{\partial W} 0 = 0
\]

**Lemma 2**: Let \( f_0, f_1 : X^k \to Y \) be smooth maps which are homotopic. Suppose also that \( \partial X^k = \emptyset \) and \( \omega \in \Omega^k(Y) \) with \( d\omega = 0 \) then:

\[
\int_X f_0^* \omega = \int_X f_1^* \omega
\]

Proof: Write \( X^k = X \) (the super-script was used to denote the dimension). Let \( F : X \times I \to Y \) be a homotopy between \( f_0 \) and \( f_1 \). It follows that \( \partial(X \times I) = (X \times \{0\}) \sqcup (X \times \{1\}) \). In addition the orientations on the boundary components must be opposite. Hence we have:

\[
0 = \int_{\partial(X \times I)} F^* \omega = \int_{X \times \{0\}} F^* \omega - \int_{X \times \{1\}} F^* \omega = \int_X f_0^* \omega + \int_X f_1^* \omega
\]
Hairy Ball Theorem

At last we are at the final result. Although the road was not completely rigorous and concise, the exploration was beautiful.

**Theorem:** There exists no non-vanishing tangent vector field on $S^2$.

**Proof:** Let $f = \text{id}_{S^2}$ and $g = -f$ i.e the antipodal map i.e $f, g : S^2 \to S^2$ which are also smooth. Suppose that there exists a nowhere vanishing tangent vector field $v$ on $S^2$.

Using this tangent vector field we can travel along an integral curve on the sphere i.e we have the existence of a homotopy between $f, g$. Let $\omega$ be the form we get by contracting the volume form $dx^1 \wedge dx^2 \wedge dx^3$ onto the radial vector. Then we have:

$$\int_{S^2} \omega = \int_{S^2} f^* \omega = \int_{S^2} g^* \omega = -\int_{S^2} f^* \omega = -\int_{S^2} \omega$$

Since the former is nonzero, we’ve reached a contradiction. We can now say that there is no way to comb all the hairs on a hairy-ball. However since there exists tangent vectors fields on $S^2$ which vanish only at 2 points or even 1 point, Krillin cannot argue that the Hairy Ball theorem is the reason for his dilemma.