AN INTEGRAL EQUATION TECHNIQUE FOR SOLVING
ELECTROMAGNETIC PROBLEMS
WITH ELECTRICALLY SMALL AND ELECTRICALLY
LARGE REGIONS

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ABSTRACT


Previously, Electric Field Integral Equations (EFIE) were derived for electromagnetic scattering problems with both electrically small and electrically large regions. The electrically small regions, or quasi-static regions, can be geometrically complex and can contain both perfect dielectrics and perfect conductors. The electrically large, or full-wave regions, contain perfect conductors in a homogeneous medium. For these particular electromagnetic problems, it was assumed that electrostatic effects were dominant in the quasi-static regions. In an effort to extend the previously developed EFIE, new integral equations were derived to include inductive effects in the quasi-static regions. This would allow the dielectrics in the quasi-static regions to have values of relative permeability other than unity. The moment method was then used to explicitly solve for the charges and currents in the newly derived EFIE. This method was then successfully validated by solving various known scattering problems.
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DEDICATION
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LIST OF SYMBOLS

\( a \) .................................................. Radius of a wire
\( \hat{a}_x \) .... Unit Vector in the \( x \)-direction defined on a Rectangular Coordinate System
\( \hat{a}_y \) .... Unit Vector in the \( y \)-direction defined on a Rectangular Coordinate System
\( \hat{a}_z \) .... Unit Vector in the \( z \)-direction defined on a Rectangular Coordinate System
\( \hat{a}_\rho \) .... Unit Vector in the \( \rho \)-direction defined on a Cylindrical Coordinate System
\( \hat{a}_\phi \) .... Unit Vector in the \( \phi \)-direction defined on a Cylindrical Coordinate System
\( \hat{a}_z \) .... Unit Vector in the \( z \)-direction defined on a Cylindrical Coordinate System
\( A \) ........................................ Magnetic Vector Potential
\( \bar{B} \) ........................................ Magnetic Flux Density
\( c \) ........................................ Speed of Light in a Vacuum
\( \Delta n, \Delta z, \Delta z', \Delta s \) .............. Distances in Central Difference Approximation
\( \nabla, \nabla' \) ........................................... Gradient
\( \bar{D} \) ........................................ Electric Flux Density
\( \varepsilon_i \) ........................................... Permittivity in the \( i^{th} \) Region
\( \varepsilon_0 \) ........................................... Permittivity of Free Space
\( \varepsilon_{r_i} \) ........................................ Relative Permittivity in the \( i^{th} \) Region
\( \bar{E} \) ......................................... Electric Field Intensity
\( \bar{E}_{tan} \) .......................................... Tangential Component of Scattered Field
\( \bar{E}_{tan} \) .......................................... Tangential Component of Incident Field
\( \bar{H} \) ......................................... Magnetic Field Intensity
\( \Rightarrow \) ........................................... “Implies”
\( \bar{I} \) ......................................... Line Current
\( \bar{I}_n \) ......................................... Unknown Line Current
\( I_n \) ......................................... Magnitude of Unknown Line Current
\( f \) ............................................ Principle Value Integral
\( \bar{J}_s \) .................................................. Surface Current Density
\( \bar{J}_n \) .................................................. Unknown Total Surface Current Density (Current Pulse)
\( k_i \) .................................................. Wave Number in the \( i \)th Region
\( \bar{k} \) .................................................. Wave Number in Direction of Propagation
\( K \) .................................................. Total Charge in Region
\( \lambda \) .................................................. Wavelength
\( \mu_i \) .................................................. Permeability in the \( i \)th Region
\( \mu_0 \) .................................................. Permeability of Free Space
\( \mu_{r_i} \) .................................................. Relative Permeability in the \( i \)th Region
\( \hat{n}_i, \hat{n}'_i \) ........................................... Unit Vector Pointing into the \( i \)th Region
\( \frac{\partial}{\partial n} \) ......................................... Partial Derivative in Normal Direction
\( N \) .................................................. Number of Segments
\( \omega \) .................................................. Source Frequency in Radians
\( \partial_i \) ............................................... Collection of all Surfaces Bounding the \( i \)th Region of Interest
\( \phi_i \) .................................................. Free Space Green’s Function
\( \psi \) .................................................. Electric Scalar Potential
\( \hat{\phi} \) ........ Unit Vector in the \( \phi \)-direction defined on a Spherical Coordinate System
\( \bar{P} \) .................................................. Polarization Vector
\( \rho_l \) .................................................. Line Charge Density
\( \rho_n \) ............................................... Unknown Total Surface Charge Density (Charge Pulse)
\( \rho_{free} \) ............................................ Free Surface Charge Density
\( \rho_{pol} \) ............................................. Polarized Surface Charge Density
\( \rho_{tot} \) ............................................ Total Surface Charge Density
\( \rho, \phi, z \) ........................................... Coordinates on a Cylindrical Coordinate System
\( \bar{r} \) .................................................. Vector Indicating a Field Point
\( \bar{r}' \) .................................................. Vector Indicating a Source Point

xv
\( \hat{R} \) ....................................................... Residual
\( R \) ......................................................... Distance Between Two Points
\( \hat{R} \) ..................................................... Unit Vector Indicating the Direction of Propagation
\( \vec{R} \) ...................................................... Vector Between Two Points
\( \sigma \) .......................................................... Conductivity
\( S_{ij} \) ........................................................ Surface Between Regions \( i \) and \( j \)
\( \sum \partial_i \) .................................................. Collection of all Boundaries in the Problem
\( \sum \partial_i^F \) .............................................. Collection of all Full-wave Boundaries in the Problem
\( \sum \partial_i^Q \) .............................................. Collection of all Quasi-static Boundaries in the Problem
\( \sum \partial_i^c \) .............................................. Conductor/Dielectric Subcollection of \( \sum \partial_i^Q \)
\( \theta_p \) ........................................................ Polarization Angle
\( \theta_i \) ........................................................ Angle of Incidence
\( \hat{\theta} \) ........ Unit Vector in the \( \theta \)-direction defined on a Spherical Coordinate System
\( \exists \) .......................................................... “There Exists”
\( u_n, v_n \) ...................................................... Unit Pulses
\( \hat{u}_n, \hat{v}_n \) ................................................ Unit Vectors
\( V_k \) .......................................................... Voltage on \( k^{th} \) Conductor
\( W_n \) .......................................................... Weighting Functions
\( \hat{z}, \hat{z}', \hat{s}, \hat{s}' \) ......................................... Tangential Unit Vector
CHAPTER 1. INTRODUCTION

Electromagnetics explains the relation between the electromagnetic fields at a point in space and their sources. Electromagnetic waves impinging on a body in space may induce surface currents on that body which then produce a scattered field. One method of determining the magnitude and phase of the scattered field is to establish and solve a set of integral equations for the unknown surface current [1]. Once these induced currents are determined, the scattered field can be evaluated. This process of establishing the appropriate integral equations can be quite difficult. So, it is reasonable to ask if there are simplifying assumptions that can be made to reduce this complexity.

Recent work [2]-[3] shows that certain simplifying assumptions are reasonable for problems with both electrically large and electrically small regions. For such problems, employing quasi-static equations in the electrically small regions results in fewer unknowns, thus reducing the complexity. This technique has been applied to problems with axial symmetry and primarily capacitively (or electrostatic) dominant electrically small regions. This technique is called the hybrid quasi-static/full-wave method [3] or hybrid method for short. The hybrid method uses a set of quasi-static equations in the electrically small regions and a set of full-wave equations in the electrically large regions. The two sets of equations are coupled together via the continuity equation.

The purpose of this thesis is to relax the constraint of assuming capacitively dominant quasi-static regions or, in other words, to include inductive effects in the quasi-static regions. To establish the foundation needed to continue the work done by Olsen, Hower, and Mannikko [2], all equations in the hybrid method were derived from first principles and implemented numerically. The derivation involved writing the electric field ($\vec{E}$) in terms of charges and currents over all surfaces. Boundary
conditions were then enforced over all interfaces in the problem. With the quasi-static assumption, relations for currents and charges were then derived. The details of this derivation are illustrated in Chapter 2. The integral equations were solved using the moment method (MOM) [4]-[5]. This involved evaluating derivatives using a central difference approximation [6] and numerical integration [6]. The details of this implementation are illustrated in Chapter 3, along with numerical results.

To extend the current hybrid method to include inductive effects in the quasi-static equations, new integral equations were derived. This extension facilitates the solution of problems containing dielectrics with values of relative permeability of unity or greater. Chapter 4 presents the complete derivation of these new integral equations. The new equations were solved using the MOM and validated by comparing the results with known scattering problems. The details of the solution method and examples are presented in Chapter 5. Concluding remarks are provided in Chapter 6.
CHAPTER 2. DERIVATION OF THE HYBRID EFIE

2.1. Introduction

This chapter presents the details involved with the derivation of the integral equations by Olsen, Hower, and Mannikko [2]. These derivations are for general electromagnetics problems \textit{without any axial symmetry constraints} and with quasi-static assumptions in certain regions. It is assumed that the quasi-static regions are capacitively dominant. Thus, inductive effects can be ignored.

2.2. Description of General Electromagnetics Problem

Consider the geometry of the general electromagnetics problem shown in Figure 1. The problem consists of regions with conductors and dielectrics. The conductors

![Diagram of General Electromagnetics Problem]

Figure 1. General Electromagnetics Problem.
are considered to have perfect conductivity ($\sigma = \infty$) and the dielectrics are linear homogeneous isotropic regions with $\sigma = 0$. $S_{12}$ denotes the smooth surface between regions 1 and 2, $S_{13}$ denotes the smooth surface between regions 1 and 3, $S_{14}$ denotes the smooth surface between regions 1 and 4, and $S_{24}$ denotes the smooth surface between regions 2 and 4. $\vec{r}$ denotes the vector measured from an arbitrary coordinate system defined on the general electromagnetics problem and $\hat{n}_i$ denotes the unit normal vector in the same coordinate system pointing into the $i^{th}$ region. The $i^{th}$ region has permittivity $\varepsilon_i = \varepsilon_r \varepsilon_0$ and permeability $\mu_i = \mu_0$; and on the subsurface $S_{i3}^s \subset S_{13}$ $\exists$ an impressed electric field representing a source in the region. The electric field in the $i^{th}$ region can be written as [2]

$$\vec{E}_i(\vec{r}) = \begin{cases} T \int_{\partial_i} \vec{Q}_i d\vec{s}' & \vec{r} \text{ in } i \\ 0 & \vec{r} \text{ not in } i \end{cases} \tag{2.1}$$

where

$$\vec{Q}_i = \frac{-1}{4\pi} \left[ j\omega \mu_0 (\hat{n}_i' \times \vec{H}_i) \phi_i - (\hat{n}_i' \times \vec{E}_i) \right] \times \nabla' \phi_i - (\hat{n}_i' \cdot \vec{E}_i) \nabla' \phi_i, \tag{2.2}$$

$\vec{H}_i$ is the magnetic field, and

$$\phi_i = \frac{e^{-jk_i R}}{R} \tag{2.3}$$

with $R = |\vec{r} - \vec{r}'|$ and $k_i = \omega \sqrt{\mu_0 \varepsilon_i}$. The fields in (2.1) are assumed to vary with $e^{j\omega t}$ and $\partial_i$ indicates that the integral is evaluated over all the smooth surfaces bounding the $i^{th}$ region of interest (note that the integral is a principle value integral [7]-[8]). The primed vectors represent the source points and the unprimed vectors represent the field points. The vector operations ($\nabla'$) and integrations ($d\vec{s}'$) are in primed coordinates, meaning these operations are performed over the source points. $T$ takes on the values of 1 and 2 in the following manner [2]: $T = 1$ if $\vec{r}$ and $\vec{r}'$ are in the same region and $T = 2$ if $\vec{r}$ is on the smooth boundary surface of the region containing $\vec{r}'$. 

4
For the remainder of this discussion it is assumed that regions 2 and 4 are electrically small and region 3 is electrically large.

Before the EFIE are derived a short discussion of the underlying principles will be presented. The Uniqueness Theorem [9] and the Equivalence Principle [9] provide the needed concepts to derive the EFIE. A version of the uniqueness theorem states that the field in a region is uniquely specified by the sources within the region plus the tangential components of the electric and magnetic (\(H\)) fields on the boundary of the region. With this in mind we can interpret (2.1) as the electric field in the \(i^{th}\) region being written uniquely in terms of the tangential components of the electric and magnetic fields on the boundary of the region. In other words, the electric field in the \(i^{th}\) region can be written uniquely in terms of the charges and currents on the boundary of the region. With this concept of the electric field the equivalence principle will be applied to each region in Figure 1. First consider region 1. An “equivalent” problem for the electric field in this region is shown in Figure 2. To support the equivalent field surface currents are defined on \(S_{12}\) and \(S_{14}\) and every region is replaced with \(\varepsilon_1\). A zero field, or null field, \((E = H = 0)\) is then defined

![Figure 2. Equivalence Theorem Illustration for Region 1.](image)
in regions 2 and 4. The incident field represents the contributions from the surface currents on $S_{13}$. A zero field is also defined in region 3; it is defined to have a permittivity of $\varepsilon_1$, but this is not illustrated in Figure 2. Figure 2 now shows us that $\vec{E}_1(\vec{r}) \neq 0$ for $\vec{r}$ in region 1 and that $\vec{E}_1(\vec{r}) = 0$ for $\vec{r}$ in any other region. This helps illustrate the conditions on $\vec{r}$ in (2.1). We can also write an “equivalent” problem for the electric field in region 2. This is shown in Figure 3. To support the equivalent field in region 2, surface currents are defined on $S_{12}$ and $S_{24}$, and every region is replaced with $\varepsilon_2$. A zero field ($\vec{E} = \vec{H} = 0$) is then defined in regions 1 and 4. A zero field is also defined in region 3, and it is defined to have a permittivity of $\varepsilon_2$. Figure 3 now shows that $\vec{E}_2(\vec{r}) \neq 0$ for $\vec{r}$ in region 2 and that $\vec{E}_2(\vec{r}) = 0$ for $\vec{r}$ in any other region. Therefore, an expression for the electric field in any region can be written as

$$\vec{E}_i(\vec{r}) = \vec{E}_1(\vec{r}) + \vec{E}_2(\vec{r})$$

(2.4)

for $i = 1, 2$. (Note: $\vec{E}_1(\vec{r}) = \vec{E}_1(\vec{r}) + 0$ for $\vec{r}$ in region 1, and $\vec{E}_2(\vec{r}) = 0 + \vec{E}_2(\vec{r})$ for $\vec{r}$

![Figure 3. Equivalence Theorem Illustration for Region 2.](image)
in region 2). This process of using the equivalence principle to write the electric field in any region in terms of the sources bounding the region can be generalized to any number of regions, implying the electric field with \( \bar{r} \) in any region can be written as

\[
\tilde{E}_i(\bar{r}) = T \int \sum \tilde{Q}_i ds'
\]

where \( \tilde{Q}_i \) is defined in (2.2) and \( \sum \partial_i \) is the collection of all boundaries in the problem. (2.5) represents the scattered field from all interfaces in the general electromagnetics problem. The previous discussion will hopefully give the needed background to derive the hybrid EFIE.

In the hybrid method the quasi-static regions are assumed to satisfy [3]

\[
|k_i^2 RJ_s| \ll |\nabla \cdot J_s|
\]

on the conductor/dielectric interfaces where \( J_s \) is the surface current density. The inequality will be satisfied if the region has a significant change in surface current density along a conductor/dielectric interface. This condition can exist at the corners of capacitor plates or at the end of a wire that is terminated in free space.

The derivations in this chapter are divided up into regions: first, quasi-static equations, followed by the full-wave equations, and finally, the current continuity equations.

2.3. Quasi-Static Equations

The first integral equations derived are for field points in the quasi-static region. A few approximations can be made if both the field points and source points are in the same quasi-static region. If \( k_iD \ll 1 \), where \( D \) is the overall maximum dimension of the quasi-static region, then \( k_iR \leq k_iD \ll 1 \). This then gives the expression

\[
e^{-j k_i R} = \cos(k_i R) - j \sin(k_i R) \approx \cos(0) - j \sin(0) = 1 \text{ for the phase term in (2.3).}
\]
Thus (2.3) can be written as
\[
\phi_i \approx \frac{1}{R}.
\] (2.6)

This greatly reduces the kernel of the EFIE in the quasi-static regions. Now \(\nabla \phi_i\) can be written as
\[
\nabla \phi_i \approx \hat{a}_x \frac{\partial}{\partial x} \frac{1}{R} + \hat{a}_y \frac{\partial}{\partial y} \frac{1}{R} + \hat{a}_z \frac{\partial}{\partial z} \frac{1}{R} = \frac{\hat{a}_x}{R^3} (x - x') + \frac{\hat{a}_y}{R^3} (y - y') + \frac{\hat{a}_z}{R^3} (z - z').
\]

\[
\Rightarrow \nabla \phi_i \approx -\frac{\bar{R}}{R^3}
\] (2.7)

where \(\bar{R} = (x - x')\hat{a}_x + (y - y')\hat{a}_y + (z - z')\hat{a}_z\) is the vector from the source point to the field point. This also gives
\[
\nabla' \phi_i \approx \hat{a}_x \frac{\partial}{\partial x'} \frac{1}{\bar{R}} + \hat{a}_y \frac{\partial}{\partial y'} \frac{1}{\bar{R}} + \hat{a}_z \frac{\partial}{\partial z'} \frac{1}{\bar{R}} = \frac{\hat{a}_x}{R^3} (x - x') + \frac{\hat{a}_y}{R^3} (y - y') + \frac{\hat{a}_z}{R^3} (z - z') = \bar{R}.
\]

\[
\Rightarrow \nabla' \phi_i = -\nabla \phi_i.
\] (2.8)

With these approximations (2.2) can be written as
\[
\bar{Q}_{i,\text{quasi}} = -\frac{1}{4\pi} \left[ j \omega \mu_0 (\hat{r} \times \bar{H}_i) \frac{1}{\bar{R}} - (\hat{r} \times \bar{E}_i) \times \frac{\bar{R}}{R^3} - (\hat{r}' \cdot \bar{E}_i) \frac{\bar{R}}{R^3} \right].
\] (2.9)

### 2.3.1. General Expression for the Electric Field

To evaluate the EFIE in the quasi-static regions a general expression for the electric field with \(\bar{r}\) in any quasi-static region needs to be determined. This expression can then be used to apply the boundary conditions on each quasi-static surface in
Figure 1. Using (2.1) the electric field in region 1 can be written as

\[ \vec{E}_1(\vec{r}) = \frac{1}{4\pi} \int_{S_{13}} \vec{Q}_1 ds' + \frac{1}{4\pi} \int_{S_{12}} \vec{Q}_{1,\text{quasi}} ds' + \frac{1}{4\pi} \int_{S_{14}} \vec{Q}_{1,\text{quasi}} ds'. \] (2.10)

Similarly, the electric field in region 2 can be written as

\[ \vec{E}_2(\vec{r}) = \frac{1}{4\pi} \int_{S_{12}} \vec{Q}_{2,\text{quasi}} ds' + \frac{1}{4\pi} \int_{S_{24}} \vec{Q}_{2,\text{quasi}} ds'. \] (2.11)

Notice (2.10) and (2.11) are written only in terms of the sources bounding the region. This then gives

\[ \vec{E}_1(\vec{r}) = \frac{1}{4\pi} \int_{S_{13}} - \frac{j \omega \mu_0 (\hat{n}_1' \times \vec{H}_1) \phi_1 - (\hat{n}_1' \times \vec{E}_1) \cdot \nabla \phi_1 - (\hat{n}_1' \cdot \vec{E}_1) \nabla' \phi_1}{R^3} ds' + \frac{1}{4\pi} \int_{S_{12}} - \frac{j \omega \mu_0 (\hat{n}_1' \times \vec{H}_1) \frac{1}{R} - (\hat{n}_1' \times \vec{E}_1) \times \frac{\vec{R}}{R^3} - (\hat{n}_1' \cdot \vec{E}_1) \frac{\vec{R}}{R^3}}{R^3} ds' + \frac{1}{4\pi} \int_{S_{14}} - \frac{j \omega \mu_0 (\hat{n}_1' \times \vec{H}_1) \frac{1}{R} - (\hat{n}_1' \times \vec{E}_1) \times \frac{\vec{R}}{R^3} - (\hat{n}_1' \cdot \vec{E}_1) \frac{\vec{R}}{R^3}}{R^3} ds' \] (2.12)

and

\[ \vec{E}_2(\vec{r}) = \frac{1}{4\pi} \int_{S_{12}} \frac{j \omega \mu_0 (\hat{n}_2' \times \vec{H}_2) \frac{1}{R} - (\hat{n}_2' \times \vec{E}_2) \times \frac{\vec{R}}{R^3} - (\hat{n}_2' \cdot \vec{E}_2) \frac{\vec{R}}{R^3}}{R^3} ds' + \frac{1}{4\pi} \int_{S_{24}} \frac{j \omega \mu_0 (\hat{n}_2' \times \vec{H}_2) \frac{1}{R} - (\hat{n}_2' \times \vec{E}_2) \times \frac{\vec{R}}{R^3} - (\hat{n}_2' \cdot \vec{E}_2) \frac{\vec{R}}{R^3}}{R^3} ds'. \] (2.13)

Using (2.12) and (2.13), the expression for the electric field, with \( \vec{r} \) in the \( i \)th region,
can be written as [10]

\[
\vec{E}_i(\vec{r}) = \vec{E}_1(\vec{r}) + \vec{E}_2(\vec{r})
\]

\[
= 1 \vec{E}_{inc} + \\
1 \int_{S_{12}} \frac{1}{4\pi} \left[ j\omega \mu_0 (\hat{n}_1' \times \hat{H}_1) \frac{1}{R} - (\hat{n}_1' \times \vec{E}_1) \times \frac{\vec{R}}{R^3} - (\hat{n}_1' \cdot \vec{E}_1) \frac{\vec{R}}{R^3} \right] ds' + \\
1 \int_{S_{12}} \frac{1}{4\pi} \left[ j\omega \mu_0 (\hat{n}_2' \times \hat{H}_2) \frac{1}{R} - (\hat{n}_2' \times \vec{E}_2) \times \frac{\vec{R}}{R^3} - (\hat{n}_2' \cdot \vec{E}_2) \frac{\vec{R}}{R^3} \right] ds' + \\
1 \int_{S_{14}} \frac{1}{4\pi} \left[ j\omega \mu_0 (\hat{n}_1' \times \hat{H}_1) \frac{1}{R} - (\hat{n}_1' \times \vec{E}_1) \times \frac{\vec{R}}{R^3} - (\hat{n}_1' \cdot \vec{E}_1) \frac{\vec{R}}{R^3} \right] ds' + \\
1 \int_{S_{24}} \frac{1}{4\pi} \left[ j\omega \mu_0 (\hat{n}_2' \times \hat{H}_2) \frac{1}{R} - (\hat{n}_2' \times \vec{E}_2) \times \frac{\vec{R}}{R^3} - (\hat{n}_2' \cdot \vec{E}_2) \frac{\vec{R}}{R^3} \right] ds' \quad (2.14)
\]

where

\[
\vec{E}_{inc} = \int_{S_{13}} \frac{1}{4\pi} \left[ j\omega \mu_0 (\hat{n}_1' \times \hat{H}_1) \phi_1 - (\hat{n}_1' \times \vec{E}_1) \times \nabla' \phi_1 - (\hat{n}_1' \cdot \vec{E}_1) \nabla' \phi_1 \right] ds' \quad (2.15)
\]

is the incident electric field contribution from \( S_{13} \). Notice that the approximation in (2.6) is not used in (2.15). This is because the source points are in the full-wave region and not in the same quasi-static region as the field points. Since \( \hat{n}_1' = -\hat{n}_2' \) on \( S_{12} \) we can group the terms in (2.14) as

\[
\vec{E}_i(\vec{r}) = \left[ \vec{E}_{inc} + \\
\frac{1}{4\pi} \int_{S_{14}} \left[ j\omega \mu_0 (\hat{n}_1' \times \hat{H}_1) \frac{1}{R} - (\hat{n}_1' \times \vec{E}_1) \times \frac{\vec{R}}{R^3} - (\hat{n}_1' \cdot \vec{E}_1) \frac{\vec{R}}{R^3} \right] ds' + \\
\frac{1}{4\pi} \int_{S_{24}} \left[ j\omega \mu_0 (\hat{n}_2' \times \hat{H}_2) \frac{1}{R} - (\hat{n}_2' \times \vec{E}_2) \times \frac{\vec{R}}{R^3} - (\hat{n}_2' \cdot \vec{E}_2) \frac{\vec{R}}{R^3} \right] ds' + \\
\frac{1}{4\pi} \int_{S_{12}} \left[ j\omega \mu_0 \hat{n}_1' \times (\hat{H}_1 - \hat{H}_2) \frac{1}{R} - \\
\hat{n}_1' \times (\vec{E}_1 - \vec{E}_2) \times \frac{\vec{R}}{R^3} - \hat{n}_1' \cdot (\vec{E}_1 - \vec{E}_2) \frac{\vec{R}}{R^3} \right] ds' \right] \quad (2.16)
\]
For source points on \( S_{14} \) and \( S_{24} \), \( \mathbf{n}_1' \times \mathbf{H}_1 \) and \( \mathbf{n}_2' \times \mathbf{H}_2 \) represent the magnetic induction effects in the quasi-static region. Since it is assumed that the electrostatic effects are dominant, \( \mathbf{n}_1' \times \mathbf{H}_1 \) and \( \mathbf{n}_2' \times \mathbf{H}_2 \) can be neglected. Since the boundaries on \( S_{13}, S_{14}, \) and \( S_{24} \) are composed of a perfect conductor and dielectric, we have

\[
\mathbf{n}_1' \times \mathbf{E}_1 = 0 \tag{2.17}
\]

and

\[
\mathbf{n}_2' \times \mathbf{E}_2 = 0, \tag{2.18}
\]

thus reducing (2.16) to

\[
\vec{E}_i(\vec{r}) = 1 \left[ \vec{E}_{inc} + \frac{-1}{4\pi} \int_{S_{14}} \left[ 0 - 0 - (\mathbf{n}_1' \cdot \mathbf{E}_1) \frac{\vec{R}}{R^3} \right] ds' + \frac{-1}{4\pi} \int_{S_{24}} \left[ 0 - 0 - (\mathbf{n}_2' \cdot \mathbf{E}_2) \frac{\vec{R}}{R^3} \right] ds' + \frac{-1}{4\pi} \int_{S_{12}} \left[ j\omega \mu_0 \mathbf{n}_1' \times (\mathbf{H}_1 - \mathbf{H}_2) \frac{1}{R} - \mathbf{n}_1' \times (\mathbf{E}_1 - \mathbf{E}_2) \times \frac{\vec{R}}{R^3} - \mathbf{n}_1' \cdot (\mathbf{E}_1 - \mathbf{E}_2) \frac{\vec{R}}{R^3} \right] ds' \right]. \tag{2.19}
\]

Also, since \( S_{12} \) is made up of two perfect dielectrics, we have

\[
\mathbf{n}_1' \times (\mathbf{H}_1 - \mathbf{H}_2) = 0, \tag{2.20}
\]

\[
\mathbf{n}_1' \times (\mathbf{E}_1 - \mathbf{E}_2) = 0, \tag{2.21}
\]

and

\[
\mathbf{n}_1' \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \mathbf{n}_1' \cdot \varepsilon_0 (\varepsilon_{r_1} \mathbf{E}_1 - \varepsilon_{r_2} \mathbf{E}_2) = 0. \tag{2.22}
\]
Using (2.22) we can write a relation for the electric field as

\[ \hat{n}'_1 \cdot \vec{E}_2 = \frac{\varepsilon_{r1}}{\varepsilon_{r2}} \hat{n}'_1 \cdot \vec{E}_1. \]  

(2.23)

Using (2.23) on \( S_{12} \) we can write

\[ -\hat{n}'_1 \cdot (\vec{E}_1 - \vec{E}_2) = -\hat{n}'_1 \cdot (\vec{E}_1 - \frac{\varepsilon_{r1}}{\varepsilon_{r2}} \vec{E}_1) \]

\[ = -\hat{n}'_1 \cdot \vec{E}_1(\frac{\varepsilon_{r2} - \varepsilon_{r1}}{\varepsilon_{r2}}). \]  

(2.24)

Substituting (2.20), (2.21), and (2.24) into (2.19) we get

\[ \vec{E}_i(\vec{r}) = 1 \left[ \vec{E}_{\text{inc}} + \frac{1}{4\pi} \int_{S_{14}} (\hat{n}'_1 \cdot \vec{E}_1) \frac{\hat{R}}{R^3} ds' + \frac{1}{4\pi} \int_{S_{24}} (\hat{n}'_2 \cdot \vec{E}_2) \frac{\hat{R}}{R^3} ds' \right. 

\[ + \frac{1}{4\pi} \frac{\varepsilon_{r2} - \varepsilon_{r1}}{\varepsilon_{r2}} \int_{S_{12}} (\hat{n}'_1 \cdot \vec{E}_1) \frac{\hat{R}}{R^3} ds' \left]. \]  

(2.25)

where

\[ \vec{E}_{\text{inc}} = \int_{S_{13}} -\frac{1}{4\pi} [j\omega \mu_0 (\hat{n}'_1 \times \vec{H}_1) \phi_1 - (\hat{n}'_1 \times \vec{E}_1) \times \nabla' \phi_1 - (\hat{n}'_1 \cdot \vec{E}_1) \nabla' \phi_1] ds'. \]  

(2.26)

Again, (2.25) is the expression of the electric field for \( \vec{r} \) in any quasi-static region written in terms of all sources on all the boundaries. It should be noted that \( \vec{E}_{\text{inc}} \) can also be used to evaluate the contribution from other quasi-static regions. As a verification, it is noted that (2.25) looks very similar to equation (11) of Daffe and Olsen [10], which was derived using Laplace’s equation for the electrostatic case.

**2.3.2. Dielectric/Dielectric Interfaces**

Using the expression for the electric field in (2.25) the boundary condition for free charge will be enforced at point \( \vec{p} \) shown in Figure 4. Using the definition of the unit normal described in Figure 4, the normal components of the electric field are
related by
\[ \hat{n}_1 \cdot (\vec{D}_1 - \vec{D}_2) = \rho_{\text{free}} \]
(2.27)

with \( \rho_{\text{free}} \) denoting the free surface charge density. Since the boundary is made up of perfect dielectrics, \( \rho_{\text{free}} = 0 \). This then gives
\[ \hat{n}_1 \cdot (\varepsilon_{r_1} \vec{E}_1 - \varepsilon_{r_2} \vec{E}_2) = 0 \]
(2.28)

for the normal components of the electric field. To evaluate this boundary condition at \( \vec{p} \) the limit [10]
\[ \lim_{\vec{r}_1 \to \vec{r}_p} \varepsilon_{r_1} \hat{n}_1 \cdot \vec{E}_1(\vec{r}_1) \]
will have to be evaluated with \( \vec{r}_1 \to \vec{r}_p \) from region 1 along the dotted normal vector.
in Figure 4. This then gives

$$\lim_{r_1 \to r_p} \varepsilon_{r_1} \hat{n}_1 \cdot \vec{E}_1(\vec{r}_1) = 2\varepsilon_{r_1} \left[ \hat{n}_1 \cdot \vec{E}_{inc}(\vec{r}_p) + \right.$$

$$\left. \hat{n}_1 \cdot \frac{1}{4\pi} \int_{S_{14}} (\hat{n}'_1 \cdot \vec{E}_1) \frac{\vec{R}_1}{R_1^3} ds' + \hat{n}_1 \cdot \frac{1}{4\pi} \int_{S_{24}} (\hat{n}'_2 \cdot \vec{E}_2) \frac{\vec{R}_2}{R_2^3} ds' + \right.$$

$$\left. \frac{-1}{4\pi} \left( \frac{\varepsilon_{r_1} - \varepsilon_{r_2}}{\varepsilon_{r_2}} \right) \lim_{r_1 \to r_p} \hat{n}_1 \cdot \int_{S_{12}} (\hat{n}'_1 \cdot \vec{E}_1) \frac{\vec{R}}{R^3} ds' \right].$$

(2.29)

Notice that (2.29) is multiplied by 2, this corresponds to $T = 2$ for $\vec{r}$ on the boundary of a region. It can be shown that [10]-[11]

$$\lim_{r_1 \to r_p} \hat{n}_1 \cdot \int_{S_{14}} (\hat{n}'_1 \cdot \vec{E}_1) \frac{\vec{R}_1}{R_1^3} ds' = 2\pi(\hat{n}_1 \cdot \vec{E}_1(\vec{r}_p)) + \hat{n}_1 \cdot \int_{S_{12}} (\hat{n}'_1 \cdot \vec{E}_1) \frac{\vec{R}_p}{R_p^3} ds'. \quad (2.30)$$

The first term on the right-hand side of (2.30) exists as a result of the discontinuity of the normal component of the electric field across the charge on the boundary. Using (2.30) in (2.29) we can write

$$\lim_{r_1 \to r_p} \varepsilon_{r_1} \hat{n}_1 \cdot \vec{E}_1(\vec{r}_1) = 2\varepsilon_{r_1} \left[ \hat{n}_1 \cdot \vec{E}_{inc}(\vec{r}_p) + \right.$$

$$\left. \hat{n}_1 \cdot \frac{1}{4\pi} \int_{S_{14}} (\hat{n}'_1 \cdot \vec{E}_1) \frac{\vec{R}_1}{R_1^3} ds' + \hat{n}_1 \cdot \frac{1}{4\pi} \int_{S_{24}} (\hat{n}'_2 \cdot \vec{E}_2) \frac{\vec{R}_2}{R_2^3} ds' + \right.$$

$$\left. \frac{-1}{4\pi} \left( \frac{\varepsilon_{r_1} - \varepsilon_{r_2}}{\varepsilon_{r_2}} \right) \left( 2\pi(\hat{n}_1 \cdot \vec{E}_1(\vec{r}_p)) \right. \right.$$

$$\left. + \hat{n}_1 \cdot \int_{S_{12}} (\hat{n}'_1 \cdot \vec{E}_1) \frac{\vec{R}_p}{R_p^3} ds' \right] \right].$$

(2.31)

where $\vec{R}_1 = \vec{r}_p - \vec{r}_{14}$, $\vec{R}_2 = \vec{r}_p - \vec{r}_{24}$, and $\vec{R}_p = \vec{r}_p - \vec{r}_{12}$. $\vec{r}_{14}$, $\vec{r}_{24}$, and $\vec{r}_{12}$ are vectors indicating the source locations on $S_{14}$, $S_{24}$, and $S_{12}$, respectively. Similarly, to evaluate
(2.28) at $\bar{p}$ the limit [10]

$$\lim_{\bar{r}_2 \to \bar{r}_p} \varepsilon_{r_2} \hat{n}_1 \cdot \bar{E}_2(\bar{r}_2)$$

will have to be evaluated with $\bar{r}_2 \to \bar{r}_p$ from region 2 along the dotted normal vector in Figure 4. This then gives

$$\lim_{\bar{r}_2 \to \bar{r}_p} \varepsilon_{r_2} \hat{n}_1 \cdot \bar{E}_2(\bar{r}_2) = 2 \varepsilon_{r_2} \left[ \hat{n}_1 \cdot \bar{E}_{inc}(\bar{r}_p) + \hat{n}_1 \cdot \frac{1}{4\pi} \int_{S_{14}} (\hat{n}'_1 \cdot \bar{E}_1) \frac{\bar{R}_1}{R_1^3} d\bar{s}' + \hat{n}_1 \cdot \frac{1}{4\pi} \int_{S_{24}} (\hat{n}'_2 \cdot \bar{E}_2) \frac{\bar{R}_2}{R_2^3} d\bar{s}' + \frac{-1}{4\pi} \left( \frac{\varepsilon_{r_1} - \varepsilon_{r_2}}{\varepsilon_{r_2}} \right) \lim_{\bar{r}_2 \to \bar{r}_p} \hat{n}_1 \cdot \int_{S_{12}} (\hat{n}'_1 \cdot \bar{E}_1) \frac{\bar{R}}{R^3} d\bar{s}' \right].$$

(2.32)

It can also be shown that [10]-[11]

$$\lim_{\bar{r}_2 \to \bar{r}_p} \hat{n}_1 \cdot \int_{S_{12}} (\hat{n}'_1 \cdot \bar{E}_1) \frac{\bar{R}}{R^3} d\bar{s}' = -2\pi (\hat{n}_1 \cdot \bar{E}_1(\bar{r}_p)) + \hat{n}_1 \cdot \int_{S_{12}} (\hat{n}'_1 \cdot \bar{E}_1) \frac{\bar{R}}{R^3} d\bar{s}'.$$  

(2.33)

Similarly, the first term on the right-hand side of (2.33) exists as a result of the discontinuity of the normal component of the electric field across the charge on the boundary. Notice the sign on the second term in (2.30) is different than the sign on the second term of (2.33). This indicates the direction of arrival at $\bar{p}$ has changed. Using (2.33) in (2.32) we can write

$$\lim_{\bar{r}_2 \to \bar{r}_p} \varepsilon_{r_2} \hat{n}_1 \cdot \bar{E}_2(\bar{r}_2) = 2 \varepsilon_{r_2} \left[ \hat{n}_1 \cdot \bar{E}_{inc}(\bar{r}_p) + \hat{n}_1 \cdot \frac{1}{4\pi} \int_{S_{14}} (\hat{n}'_1 \cdot \bar{E}_1) \frac{\bar{R}_1}{R_1^3} d\bar{s}' + \hat{n}_1 \cdot \frac{1}{4\pi} \int_{S_{24}} (\hat{n}'_2 \cdot \bar{E}_2) \frac{\bar{R}_2}{R_2^3} d\bar{s}' + \frac{-1}{4\pi} \left( \frac{\varepsilon_{r_1} - \varepsilon_{r_2}}{\varepsilon_{r_2}} \right) (\hat{n}_1 \cdot \bar{E}_1(\bar{r}_p)) \right. \left. + \hat{n}_1 \cdot \int_{S_{12}} (\hat{n}'_1 \cdot \bar{E}_1) \frac{\bar{R}}{R^3} d\bar{s}' \right]$$

$$= \varepsilon_{r_2} \hat{n}_1 \cdot \bar{E}_2(\bar{r}_p)$$

(2.34)
where, again, $\bar{R}_1 = \bar{r}_p - \bar{r}_{14}$, $\bar{R}_2 = \bar{r}_p - \bar{r}_{24}$, and $\bar{R}_p = \bar{r}_p - \bar{r}_{12}$. To enforce the boundary condition on $S_{12}$, (2.34) is subtracted from (2.31). This then gives

$$\hat{n}_1 \cdot (\varepsilon_{r_1} \bar{E}_1(\bar{r}_p) - \varepsilon_{r_2} \bar{E}_2(\bar{r}_p)) = 0$$

$$= 2 \left[ \hat{n}_1 \cdot (\varepsilon_{r_1} - \varepsilon_{r_2}) \bar{E}_{inc}(\bar{r}_p) + \hat{n}_1 \cdot \frac{1}{4\pi} (\varepsilon_{r_1} - \varepsilon_{r_2}) \int_{S_{14}} (\hat{n}_1' \cdot \bar{E}_1) \frac{\bar{R}_1}{R_1^3} ds' + \hat{n}_1 \cdot \frac{1}{4\pi} \int_{S_{24}} (\hat{n}_1' \cdot \bar{E}_2) \frac{\bar{R}_2}{R_2^3} ds' \right. $$

$$- \frac{2\pi}{4\pi} \left( \frac{\varepsilon_{r_1} - \varepsilon_{r_2}}{\varepsilon_{r_2}} \right) (\varepsilon_{r_1} + \varepsilon_{r_2}) (\hat{n}_1 \cdot \bar{E}_1(\bar{r}_p)) + \left. \frac{-1}{4\pi} \left( \frac{\varepsilon_{r_1} - \varepsilon_{r_2}}{\varepsilon_{r_2}} \right) (\varepsilon_{r_1} - \varepsilon_{r_2}) \hat{n}_1 \cdot \int_{S_{12}} (\hat{n}_1' \cdot \bar{E}_1) \frac{\bar{R}_p}{R_p^3} ds' \right] .$$

Factoring out a $(\varepsilon_{r_1} - \varepsilon_{r_2})$, we get the following:

$$2(\varepsilon_{r_1} - \varepsilon_{r_2}) \left[ \hat{n}_1 \cdot \bar{E}_{inc}(\bar{r}_p) + \hat{n}_1 \cdot \frac{1}{4\pi} \int_{S_{14}} (\hat{n}_1' \cdot \bar{E}_1) \frac{\bar{R}_1}{R_1^3} ds' + \hat{n}_1 \cdot \frac{1}{4\pi} \int_{S_{24}} (\hat{n}_1' \cdot \bar{E}_2) \frac{\bar{R}_2}{R_2^3} ds' \right. $$

$$- \left( \frac{\varepsilon_{r_1} + \varepsilon_{r_2}}{2\varepsilon_{r_2}} \right) (\hat{n}_1 \cdot \bar{E}_1(\bar{r}_p)) + \left. \frac{-1}{4\pi} \left( \frac{\varepsilon_{r_1} - \varepsilon_{r_2}}{\varepsilon_{r_2}} \right) \hat{n}_1 \cdot \int_{S_{12}} (\hat{n}_1' \cdot \bar{E}_1) \frac{\bar{R}_p}{R_p^3} ds' \right] = 0 .$$

Now, solving for $\bar{E}_{inc}(\bar{r}_p)$ gives

$$\hat{n}_1 \cdot \bar{E}_{inc}(\bar{r}_p) = \hat{n}_1 \cdot \frac{-1}{4\pi} \int_{S_{14}} (\hat{n}_1' \cdot \bar{E}_1) \frac{\bar{R}}{R^3} ds' + \hat{n}_1 \cdot \frac{-1}{4\pi} \int_{S_{24}} (\hat{n}_1' \cdot \bar{E}_2) \frac{\bar{R}}{R^3} ds' + \left( \frac{\varepsilon_{r_1} + \varepsilon_{r_2}}{2\varepsilon_{r_2}} \right) (\hat{n}_1 \cdot \bar{E}_1(\bar{r}_p)) + \left( \frac{\varepsilon_{r_1} - \varepsilon_{r_2}}{4\pi\varepsilon_{r_2}} \right) \hat{n}_1 \cdot \int_{S_{12}} (\hat{n}_1' \cdot \bar{E}_1) \frac{\bar{R}}{R^3} ds' .$$

(2.35)

Notice that the subscripts for $R$ and $\bar{R}$ have been dropped. As a verification, it
is noted that (2.35) is exactly the same as equation (9) in [2]. (2.35) can now be written in terms of total surface charge density ($\rho_{tot}$) and surface current density ($\vec{J}_s$) only. First, several expressions for $\vec{J}_s$ and $\rho_{tot}$ need to be evaluated. Since $S_{12}$ is a dielectric/dielectric interface, the normal components of the electric field are related through

$$\hat{n}_1 \cdot [\varepsilon_1 \vec{E}_1(\vec{r}) - \varepsilon_2 \vec{E}_2(\vec{r})] = \rho_{free}(\vec{r})$$  \hspace{1cm} (2.36)$$

and

$$-\hat{n}_1 \cdot [\vec{P}_1(\vec{r}) - \vec{P}_2(\vec{r})] = \rho_{pol}(\vec{r})$$  \hspace{1cm} (2.37)$$

where $\vec{P}_1 = \varepsilon_0 \chi \vec{E}_1(\vec{r}) = \varepsilon_0 (\varepsilon_{r_1} - 1) \vec{E}_1(\vec{r})$ [12]. Again, $\rho_{free}$ denotes the free surface charge density; $\rho_{pol}$ denotes the polarized surface charge density; and $\vec{P}$ is the polarization vector. Since $\rho_{free} = 0$ on $S_{12}$, we can solve for the electric field in region 2 as

$$\hat{n}_1 \cdot \vec{E}_2(\vec{r}) = \frac{\varepsilon_{r_1}}{\varepsilon_{r_2}} \hat{n}_1 \cdot \vec{E}_1(\vec{r}).$$  \hspace{1cm} (2.38)$$

Using (2.37) and (2.38), the following relation for $\rho_{pol}$ can be written

$$\rho_{pol}(\vec{r}) = -\hat{n}_1 \cdot (\vec{P}_1(\vec{r}) - \vec{P}_2(\vec{r}))$$

$$= -\varepsilon_0 \hat{n}_1 \cdot (\varepsilon_{r_1} - 1) \vec{E}_1(\vec{r}) - (\varepsilon_{r_2} - 1) \vec{E}_2(\vec{r}))$$

$$= -\varepsilon_0 \hat{n}_1 \cdot \left(\varepsilon_{r_1} - 1\right) \vec{E}_1(\vec{r}) - (\varepsilon_{r_2} - 1) \frac{\varepsilon_{r_1}}{\varepsilon_{r_2}} \vec{E}_1(\vec{r})$$

$$= \varepsilon_0 \left(1 - \frac{\varepsilon_{r_1}}{\varepsilon_{r_2}}\right) \hat{n}_1 \cdot \vec{E}_1(\vec{r})$$  \hspace{1cm} (2.39)$$

$$\Rightarrow \left(\frac{\varepsilon_{r_2} - \varepsilon_{r_1}}{\varepsilon_{r_2}}\right) \hat{n}_1 \cdot \vec{E}_1(\vec{r}) = \frac{\rho_{pol}(\vec{r})}{\varepsilon_0} = \frac{\rho_{tot}(\vec{r})}{\varepsilon_0}.$$  \hspace{1cm} (2.40)$$

Next, on the conductor/dielectric interfaces of $S_{14}$ the normal components of the
electric field are related to the charge by

\[ \rho_{\text{tot}}(\vec{r}) = \rho_{\text{free}}(\vec{r}) + \rho_{\text{pol}}(\vec{r}) \]

\[ = \hat{n}_1 \cdot [\varepsilon_0 \varepsilon_{r_1} \bar{E}_1(\vec{r}) - \varepsilon_0 \varepsilon_{r_2} \bar{E}_4(\vec{r})] - \hat{n}_1 \cdot (\bar{P}_1(\vec{r}) - \bar{P}_4(\vec{r})) \]

\[ = \hat{n}_1 \cdot [\varepsilon_0 \varepsilon_{r_1} \bar{E}_1(\vec{r}) - 0] - \hat{n}_1 \cdot (\bar{P}_1(\vec{r}) - 0) \]

\[ = \hat{n}_1 \cdot \bar{E}_1(\vec{r})[\varepsilon_0 \varepsilon_{r_1} - \varepsilon_0(\varepsilon_{r_1} - 1)] \]

\[ = \hat{n}_1 \cdot \bar{E}_1(\vec{r}) \varepsilon_0 \]

(2.41)

\[ \Rightarrow \]

\[ \hat{n}_1 \cdot \bar{E}_1(\vec{r}) = \frac{\rho_{\text{free}}(\vec{r}) + \rho_{\text{pol}}(\vec{r})}{\varepsilon_0} = \frac{\rho_{\text{tot}}(\vec{r})}{\varepsilon_0}. \]

(2.42)

Also on the conductor/dielectric interfaces of \( S_{13} \) we have the following relations for the tangential components of the electric and magnetic fields:

\[ \hat{n}_1 \times \bar{E}_1(\vec{r}) = 0 \]

(2.43)

and

\[ \hat{n}_1 \times \bar{H}_1(\vec{r}) = \bar{J}_s. \]

(2.44)

Now, first rearranging (2.35) gives

\[
\left( \frac{\varepsilon_{r_1} + \varepsilon_{r_2}}{2\varepsilon_{r_2}} \right) \left( \frac{\varepsilon_{r_2} - \varepsilon_{r_1}}{\varepsilon_{r_2} - \varepsilon_{r_1}} \right) (\hat{n}_1 \cdot \bar{E}_1(\vec{r}_p)) = \hat{n}_1 \cdot \bar{E}_{\text{inc}}(\vec{r}_p) +
\]

\[ \hat{n}_1 \cdot \frac{1}{4\pi} \int_{S_{14}} (\hat{n}_1' \cdot \bar{E}_1) \frac{\bar{R}}{R^3} ds' + \]

\[ \hat{n}_1 \cdot \frac{1}{4\pi} \int_{S_{24}} (\hat{n}_2' \cdot \bar{E}_2) \frac{\bar{R}}{R^3} ds' + \]

\[ \hat{n}_1 \cdot \left( \frac{\varepsilon_{r_2} - \varepsilon_{r_1}}{4\pi\varepsilon_{r_2}} \right) \int_{S_{12}} (\hat{n}_1' \cdot \bar{E}_1) \frac{\bar{R}}{R^3} ds'. \]
Multiplying both sides of (2.45) by \((\varepsilon_{r_2} - \varepsilon_{r_1})\) and substituting (2.40) and (2.42) gives

\[
\left(\frac{\varepsilon_{r_1} + \varepsilon_{r_2}}{2\varepsilon_0}\right)\rho_{tot}(\vec{r}_p) = - (\varepsilon_{r_1} - \varepsilon_{r_2})\hat{n}_1 \cdot \vec{E}_{inc}(\vec{r}_p) -
\]

\[
\frac{(\varepsilon_{r_1} - \varepsilon_{r_2})}{4\pi}\hat{n}_1 \cdot \int_{S_{14}} \frac{\rho_{tot}(\vec{r}')}{\varepsilon_0} \frac{\vec{R}}{\vec{R}^3} d\vec{s}' -
\]

\[
\frac{(\varepsilon_{r_1} - \varepsilon_{r_2})}{4\pi}\hat{n}_1 \cdot \int_{S_{24}} \frac{\rho_{tot}(\vec{r}')}{\varepsilon_0} \frac{\vec{R}}{\vec{R}^3} d\vec{s}' -
\]

\[
\frac{(\varepsilon_{r_1} - \varepsilon_{r_2})}{4\pi}\hat{n}_1 \cdot \int_{S_{12}} \frac{\rho_{tot}(\vec{r}')}{\varepsilon_0} \frac{\vec{R}}{\vec{R}^3} d\vec{s}'.
\]  

(2.46)

Using (2.42), (2.43), (2.44), and the current continuity equation \(-j\omega \rho_s = \nabla' \cdot \vec{J}_s\), where \(\rho_s\) is the surface charge density, \(\vec{E}_{inc}(\vec{r}_p)\) can be written as

\[
\vec{E}_{inc}(\vec{r}_p) = \frac{-1}{4\pi} \int_{S_{13}} j\omega \mu_0 \vec{J}_s(\vec{r}') \phi_1 d\vec{s}' - \frac{-1}{4\pi j\omega \varepsilon_0} \int_{S_{13}} \nabla' \cdot \vec{J}_s(\vec{r}') \nabla' \phi_1 d\vec{s}'.
\]  

(2.47)

Using \(\nabla \phi_1 = -\nabla \phi'_1\) we can write the expression for the incident electric field as

\[
\vec{E}_{inc}(\vec{r}_p) = \frac{-1}{4\pi} \int_{S_{13}} j\omega \mu_0 \vec{J}_s(\vec{r}') \phi_1 d\vec{s}' + \nabla \frac{1}{4\pi j\omega \varepsilon_0} \int_{S_{13}} \nabla' \cdot \vec{J}_s(\vec{r}') \phi_1 d\vec{s}'.
\]  

(2.48)

(2.46) can be written in the following compact form:

\[
\left(\frac{\varepsilon_{r_1} + \varepsilon_{r_2}}{2\varepsilon_0}\right)\rho_{tot}(\vec{r}_p) = - (\varepsilon_{r_1} - \varepsilon_{r_2})\hat{n}_1 \cdot \vec{E}_{inc}(\vec{r}_p) -
\]

\[
\frac{(\varepsilon_{r_1} - \varepsilon_{r_2})}{4\pi}\hat{n}_1 \cdot \int_{\sum \partial_i^Q} \frac{\rho_{tot}(\vec{r}')}{\varepsilon_0} \frac{\vec{R}}{\vec{R}^3} d\vec{s}'.
\]  

(2.49)

where \(\sum \partial_i^Q\) is the collection of all quasi-static surfaces. Then, in a slightly more general form, we get

\[
\left(\frac{\varepsilon_{r_i} + \varepsilon_{r_j}}{2\varepsilon_0}\right)\rho_{tot}(\vec{r}_p) = - (\varepsilon_{r_i} - \varepsilon_{r_j})\hat{n}_i \cdot \vec{E}_{inc}(\vec{r}_p) -
\]

\[
\frac{(\varepsilon_{r_i} - \varepsilon_{r_j})}{4\pi}\hat{n}_i \cdot \int_{\sum \partial_i^Q} \frac{\rho_{tot}(\vec{r}')}{\varepsilon_0} \frac{\vec{R}}{\vec{R}^3} d\vec{s}'.
\]  

(2.50)
where again

\[
\tilde{E}_{inc}(\bar{r}_p) = \frac{-1}{4\pi} \int_{S_{13}} j\omega\mu_0 \tilde{J}_s(\bar{r}')\phi_1 d\mathbf{s}' + \nabla \frac{1}{4\pi j\omega\varepsilon_0} \int_{S_{13}} \nabla' \cdot \tilde{J}_s(\bar{r})\phi_1 d\mathbf{s}'.
\]  
(2.51)

(2.50) enforces the boundary condition on \( S_{12} \) and is written in terms of all the sources on all the boundaries. As another verification, note that (2.50) is exactly like equation (2) of [3]. This is the first equation with two unknowns: charge and current. This now completes our derivation for the dielectric/dielectric boundary condition on \( S_{12} \).

2.3.3. Conductor/Dielectric Interfaces

The next boundary condition enforced in Figure 1 will be on the surfaces \( S_{14} \) and \( S_{24} \). These surfaces consist of a conductor/dielectric interface. Choosing \( S_{14} \), the boundary condition for the electric field is

\[
\hat{n}_1 \times \tilde{E}_1(\bar{r}_1) = 0
\]  
(2.52)

where \( \bar{r}_1 \) is the position vector for a field point on \( S_{14} \). Using (2.25) in (2.52) gives

\[
\hat{n}_1 \times \tilde{E}_1(\bar{r}) = 0 = \hat{n}_1 \times 2 \left[ \tilde{E}_{inc} + \frac{1}{4\pi} \int_{S_{14}} (\hat{n}'_1 \cdot \tilde{E}_1) \frac{\bar{R}}{R'_{12}} d\mathbf{s}' + \frac{1}{4\pi} \int_{S_{24}} (\hat{n}'_2 \cdot \tilde{E}_2) \frac{\bar{R}}{R'_{24}} d\mathbf{s}' + \frac{1}{4\pi} \int_{S_{12}} (\hat{n}'_1 \cdot \tilde{E}_1) \frac{\bar{R}}{R'_{12}} d\mathbf{s}' \right].
\]  
(2.53)

From (2.7) we can write

\[
\hat{n}_1 \times \tilde{E}_1(\bar{r}) = 0 = \hat{n}_1 \times 2 \left[ \tilde{E}_{inc} + \nabla \frac{-1}{4\pi} \int_{S_{14}} (\hat{n}'_1 \cdot \tilde{E}_1) \frac{1}{\bar{R}} d\mathbf{s}' + \nabla' \frac{-1}{4\pi} \int_{S_{24}} (\hat{n}'_2 \cdot \tilde{E}_2) \frac{1}{\bar{R}} d\mathbf{s}' + \nabla \frac{-1}{4\pi} \int_{S_{12}} (\hat{n}'_1 \cdot \tilde{E}_1) \frac{1}{\bar{R}} d\mathbf{s}' \right].
\]

For field points in the quasi-static region it has been shown [2] that \( \tilde{E}_{inc} \approx -\nabla \psi_{inc}(\bar{r}) \).
This then gives
\[ \hat{n}_1 \times \bar{E}_1(\bar{r}) = \hat{n}_1 \times \nabla \chi = 0 \quad (2.54) \]

where
\[
\chi = -\psi_{\text{inc}} + \frac{1}{4\pi} \int_{S_{14}} (\hat{n}'_1 \cdot \bar{E}_1) \frac{1}{R} ds' + \frac{1}{4\pi} \int_{S_{24}} (\hat{n}'_2 \cdot \bar{E}_2) \frac{1}{R} ds' + \frac{1}{4\pi} \frac{\varepsilon_{r_1} - \varepsilon_{r_2}}{\varepsilon_{r_2}} \int_{S_{12}} (\hat{n}'_1 \cdot \bar{E}_1) \frac{1}{R} ds',
\]
(2.55)

and from (2.15),
\[
\bar{E}_{\text{inc}} = -\frac{1}{4\pi} \int_{S_{13}} j \omega \mu_0 (\hat{n}'_1 \times \bar{H}_1) \phi_1 ds' - \nabla \frac{1}{4\pi} \int_{S_{13}} (\hat{n}'_1 \cdot \bar{E}_1) \phi_1 ds'.
\]
(2.56)

Since \( S_{14} \) is an interface between a dielectric and perfect conductor, (2.54) is zero everywhere. Note that (2.54) states that, along the surface of the conductor, \( \chi \) has no spacial variation. Therefore,
\[ \chi = V \quad (2.57) \]

which is an expression indicating that a constant voltage exists on all conductors, or in other words, \( S_{14} \) is an equipotential surface. As another verification, note that (2.54)-(2.57) are exactly like equations (11)-(13) derived in [2]. Equation (2.54) can be written in terms of charges and currents only. Using (2.40), (2.42), and multiplying both sides of (2.54) by \(-1\), we can write (2.55) as
\[
\chi = \psi_{\text{inc}}(\bar{r}) + \frac{1}{4\pi \varepsilon_0} \left[ \int_{S_{14}} \frac{\rho_{\text{tot}}(\bar{r}')}{R} ds' + \int_{S_{24}} \frac{\rho_{\text{tot}}(\bar{r}')}{R} ds' + \int_{S_{12}} \frac{\rho_{\text{tot}}(\bar{r}')}{R} ds' \right].
\]
(2.58)

Equation (2.58) can be written in a slightly more compact form of
\[
\chi = \psi_{\text{inc}}(\bar{r}) + \frac{1}{4\pi \varepsilon_0} \int_{\sum \partial_{i}^Q} \frac{\rho_{\text{tot}}(\bar{r}')}{R} ds'.
\]
(2.59)
Using $\hat{n}'_1 \times \vec{H}_1 = \vec{J}_s$ and $-j\omega \rho_s = \nabla' \cdot \vec{J}_s$ on $S_{13}$, it can be shown [2] that $\psi_{inc}(\vec{r})$ can be determined within a constant term as

$$
\psi_{inc} = (\vec{r} - \vec{r}_0) \frac{1}{4\pi} \int_{S_{13}} j\omega \mu_0 \vec{J}_s(\vec{r}') \tilde{\phi}_1 d\vec{s}' - \frac{1}{4\pi j\omega \varepsilon_0} \int_{S_{13}} \nabla' \cdot \vec{J}_s(\vec{r}') \phi_1 d\vec{s}'
$$

(2.60)

where $\tilde{\phi}_1 = \frac{e^{-jkR_0}}{R_0}$ and $R_0$ is the distance from the source point to the center of the quasi-static region. $(\vec{r} - \vec{r}_0)$ is the vector from the match point to the center of the quasi-static region. Recalling from (2.57) that $\chi$ is a constant, the unknown constant involved in the determination of $\psi_{inc}$ can be included in another constant $V_k$ such that

$$
V_k = \psi_{inc}(\vec{r}) + \frac{1}{4\pi \varepsilon_0} \int \sum \frac{\rho_{tot}(\vec{r}')}{R} d\vec{s}'
$$

(2.61)

where $V_k$ is the voltage on the $k^{th}$ conductor. (2.61) enforces the boundary condition on $S_{14}$ and is written in terms of all sources on all the boundaries. Note that (2.61) is exactly like equation (3) in [3]. (2.50) and (2.61) gives us two equations with three unknowns. The first two unknowns are the charges and currents in the quasi-static region, and the third unknown is the current in the full-wave region.

2.4. Full-Wave Equations

The derivations for the full-wave EFIE are very similar to the steps arriving to (2.54). The major difference is that quasi-static assumptions cannot be assumed here.

2.4.1. General Expression for the Electric Field

Using the equivalence principle, the expression for the electric field in region 1 can be written as

$$
\vec{E}_1(\vec{r}) = -\frac{1}{4\pi} \int_{S_{13}} j\omega \mu_0 (\hat{n}'_1 \times \vec{H}_1) \phi_1 d\vec{s}' + \frac{1}{4\pi} \int_{S_{13}} (\hat{n}'_1 \cdot \vec{E}_1) \nabla' \phi_1 d\vec{s}' + \vec{E}^Q(\rho, z)
$$

(2.62)

where $\vec{E}^Q(\rho, z)$ is the expression for the electric field from sources on $S_{12}, S_{14},$ and $S_{24}$.
in the quasi-static region. The expression for the electric field from these quasi-static sources can be written as

\[
\vec{E}^Q(\rho, s) = -j\omega \vec{A}_1 - \nabla \psi_1
\]  

(2.63)

with

\[
\vec{A}_1 = \frac{1}{4\pi} \int_{\Sigma} \mu_0 (\hat{n}'_i \times \vec{H}_i) \phi_1 ds',
\]

(2.64)

\[
\psi_1 = \frac{1}{4\pi} \int_{\Sigma} (\hat{n}'_i \cdot \vec{E}_i) \phi_1 ds',
\]

(2.65)

and \(i=1,2\).

2.4.2. Conductor/Dielectric Interfaces

Now the boundary condition \(\hat{n}_1 \times \vec{E}_1 = 0\) is enforced on \(S_{13}\) in Figure 1. Using (2.62), the following expression for the electric field can be written as [2]

\[
\hat{n}_1 \times \vec{E}_1(\vec{r}) = 0
\]

(2.67)

Now (2.66) can be written in terms of charge and current only. Using (2.42), the current continuity equation \(-j\omega \rho_s = \nabla' \cdot \vec{J}_s\), and \((\hat{n}'_1 \times \vec{H}_1) = \vec{J}_s\), (2.66) can be written as

\[
\hat{n}_1 \times \vec{E}_1(\vec{r}) = 0
\]  

(2.67)
where

\[
\bar{E}_1(\bar{r}) = \frac{-1}{4\pi} \int_{S_{13}} j\omega \mu_0 \bar{J}_s(\bar{r}') \phi_1 ds' + \nabla \frac{1}{4\pi j\omega \varepsilon_0} \int_{S_{13}} \nabla' \cdot \bar{J}_s(\bar{r}) \phi_1 ds' + \bar{E}^Q(\rho, s) \tag{2.68}
\]

and

\[
\bar{E}^Q(\rho, s) = \frac{-1}{4\pi} \int_{\sum \partial^Q_i} j\omega \mu_0 \bar{J}_s(\bar{r}') \phi_1 ds' - \nabla \frac{1}{4\pi \varepsilon_0} \int_{\sum \partial^Q_i} \rho_s(\bar{r}) \phi_1 ds' \tag{2.69}
\]

for \(i=1,2\). (2.67) is written in terms of all sources on all boundaries, this then gives us a third equation in terms of two unknowns: charge and current. This ends the derivation for the EFIE on the conductor/dielectric interface in the full-wave region. All types of boundary conditions in the full-wave region are now evaluated and only one more equation is needed to provide enough equations to solve for the unknown charges and currents.

### 2.5. Current Continuity Derivations

The next equation considered is a combination of the current continuity equation and the conservation of charge. This equation relates the charges in both regions to currents in both regions. The conservation of charge is enforced through

\[
\varepsilon_{r_i} \int_{\sum \partial^F_i} \rho_s(\bar{r}') ds' + \varepsilon_{r_i} \int_{\sum \partial^Q_i} \rho_s(\bar{r}') ds' = K \tag{2.70}
\]

and the current continuity, in general, is

\[
\nabla' \cdot \bar{J}_s(\bar{r}') = -j\omega \rho_s(\bar{r}') \tag{2.71}
\]

where \(\sum \partial^F_i\) is the collection of all boundaries in the full-wave region. \(K\) is the total charge in a region and is assumed to be zero for our problems. Since the only unknown in the full-wave region is current, (2.70) will be put in terms of both current...
and charge. By multiplying both sides of (2.70) by \( j\omega \) and substituting (2.71) for the first term we get

\[
\tilde{J}(\tilde{r}_a) - \tilde{J}(\tilde{r}_b) = -j\omega \varepsilon_{r_i} \int_{\partial Q^2} \rho_s(\tilde{r}') ds' 
\]

(2.72)

where \( \tilde{J} \) is the current along the full-wave surface between the limits of integration \( \tilde{r}_a \) and \( \tilde{r}_b \). (2.72) gives the final relation needed between current and charge in both the quasi-static region and full-wave regions.

2.6. Summary of Hybrid EFIE

Before the hybrid EFIE are evaluated a summary of the derived equations would be helpful. For dielectric/dielectric interfaces in the quasi-static region we have

\[
\left( \frac{\varepsilon_{r_i} + \varepsilon_{r_j}}{2\varepsilon_0} \right) \rho_{\text{tot}}(\tilde{r}_p) = -(\varepsilon_{r_i} - \varepsilon_{r_j}) \hat{n}_i \cdot \tilde{E}_{\text{inc}}(\tilde{r}_p) - \frac{(\varepsilon_{r_i} - \varepsilon_{r_j})}{4\pi} \hat{n}_i \cdot \int_{\partial Q^2} \frac{\rho_{\text{tot}}(\tilde{r}') \tilde{R}}{\varepsilon_0 R^3} ds' 
\]

(2.73)

where

\[
\tilde{E}_{\text{inc}}(\tilde{r}_p) = -\frac{1}{4\pi} \int_{S_{13}} j\omega \mu_0 \tilde{J}_s(\tilde{r}') \phi_1 ds' + \nabla \frac{1}{4\pi j\omega \varepsilon_0} \int_{S_{13}} \nabla' \cdot \tilde{J}_s(\tilde{r}') \phi_1 ds'. 
\]

(2.74)

For conductor/dielectric interfaces in the quasi-static region we have

\[
V_k = \psi_{\text{inc}}(\tilde{r}) + \frac{1}{4\pi \varepsilon_0} \int_{\partial Q^2} \frac{\rho_{\text{tot}}(\tilde{r}')}{\tilde{R}} ds' 
\]

(2.75)

where

\[
\psi_{\text{inc}} = (\tilde{r} - \tilde{r}_0) \frac{1}{4\pi} \int_{S_{13}} j\omega \mu_0 \tilde{J}_s(\tilde{r}') \phi_1 ds' - \frac{1}{4\pi j\omega \varepsilon_0} \int_{S_{13}} \nabla' \cdot \tilde{J}_s(\tilde{r}') \phi_1 ds'. 
\]

(2.76)
For conductor/dielectric interfaces in the full-wave region we have

\[ \hat{n}_1 \times \vec{E}_1(\bar{r}) = 0 \]  

(2.77)

where

\[ \vec{E}_1(\bar{r}) = \frac{-1}{4\pi} \int_{S_{13}} j\omega\mu_0 \vec{J}_s(\bar{r}')\phi_1 ds' + \nabla \frac{1}{4\pi j\omega\varepsilon_0} \int_{S_{13}} \nabla' \cdot \vec{J}_s(\bar{r}')\phi_1 ds' + \vec{E}^Q(\rho, s) \]  

(2.78)

and

\[ \vec{E}^Q(\rho, s) = \frac{-1}{4\pi} \int_{\Sigma_{\rho}} j\omega\mu_0 \vec{J}_s(\bar{r}')\phi_1 ds' - \nabla \frac{1}{4\pi\varepsilon_0} \int_{\Sigma_{\rho}} \rho_s(\bar{r}')\phi_1 ds' \]  

(2.79)

for \( i = 1, 2 \). Finally, the current continuity equations for both regions are given as

\[ \vec{J}(\bar{r}_a) - \vec{J}(\bar{r}_b) = -j\omega \varepsilon_{r_i} \int_{\Sigma_{\rho}} \rho_s(\bar{r}') ds'. \]  

(2.80)

(2.73)-(2.80) give enough equations for the unknown charges and currents.
CHAPTER 3. IMPLEMENTATION OF HYBRID EFIE

3.1. The Moment Method

The moment method was used to solve the derived integral equations in the hybrid method. The moment method is a linear algebra technique to approximate a solution to [13]

\[ L(f) = g \]  \hspace{1cm} (3.1)

where \( f \) is an unknown function, \( L \) is a linear operator, and \( g \) is a forcing function.

To solve (3.1), \( f \) is written in terms of known expansion functions denoted by \( f_n \) and an unknown amplitude denoted by \( \alpha_n \). This then gives

\[ f \approx \tilde{f} = \sum_{n=1}^{N} f_n \alpha_n. \]  \hspace{1cm} (3.2)

The expansion functions \( f_n \) are linearly independent in the domain of the linear operator. The approximation in (3.2) introduces an error between the exact solution and the approximate solution. This error is denoted by the residual \( \tilde{R} \) and is written as

\[ \tilde{R} = L(f) - L(\tilde{f}). \]  \hspace{1cm} (3.3)

Substituting (3.1) into (3.3), we can then write

\[ \tilde{R} = g - L(\tilde{f}). \]  \hspace{1cm} (3.4)

To minimize \( \tilde{R} \), the inner product (\( \langle *, * \rangle \)) will be taken with a set of known weighting functions, \( W_n \). \( W_n \) is defined in the range of the linear operator. Using these weighting functions and (3.4) in the inner product, we can minimize \( \tilde{R} \) by letting

\[ \langle W_m, \tilde{R} \rangle = 0 \]  \hspace{1cm} (3.5)
for \( m = 1, 2, \ldots, N \). Using (3.2) and (3.4) in (3.5), we can then write

\[
\langle W_m, g - \sum_{n=1}^{N} \alpha_n L(f_n) \rangle = 0.
\] (3.6)

Rewriting (3.6) we get

\[
\sum_{n=1}^{N} \alpha_n \langle W_m, L(f_n) \rangle = \langle W_m, g \rangle
\] (3.7)

for \( m = 1, 2, \ldots, N \). Equation (3.7) can also be written in matrix form in the following manner:

\[
\begin{bmatrix}
\langle W_1, L(f_1) \rangle & \langle W_1, L(f_2) \rangle & \cdots & \langle W_1, L(f_N) \rangle \\
\langle W_2, L(f_1) \rangle & \langle W_2, L(f_2) \rangle & \cdots & \langle W_2, L(f_N) \rangle \\
\vdots & \vdots & \ddots & \vdots \\
\langle W_N, L(f_1) \rangle & \langle W_N, L(f_2) \rangle & \cdots & \langle W_N, L(f_N) \rangle 
\end{bmatrix}
\begin{bmatrix}
\alpha_1 \\
\alpha_2 \\
\vdots \\
\alpha_N 
\end{bmatrix}
= 
\begin{bmatrix}
\langle W_1, g \rangle \\
\langle W_2, g \rangle \\
\vdots \\
\langle W_N, g \rangle 
\end{bmatrix}.
\] (3.8)

The matrix in (3.8) only contains the unknowns \( \alpha_n \) because \( W_n \) were defined weighting functions, \( f_n \) were defined expansion functions, and the forcing function \( g \) is a known condition. To solve for the unknown \( \alpha_n \)'s, we can multiply both sides of (3.8) by the inverse of the left most matrix in (3.8). This then gives

\[
\begin{bmatrix}
\alpha_1 \\
\alpha_2 \\
\vdots \\
\alpha_N 
\end{bmatrix} = 
\begin{bmatrix}
\langle W_1, L(f_1) \rangle & \langle W_1, L(f_2) \rangle & \cdots & \langle W_1, L(f_N) \rangle \\
\langle W_2, L(f_1) \rangle & \langle W_2, L(f_2) \rangle & \cdots & \langle W_2, L(f_N) \rangle \\
\vdots & \vdots & \ddots & \vdots \\
\langle W_N, L(f_1) \rangle & \langle W_N, L(f_2) \rangle & \cdots & \langle W_N, L(f_N) \rangle 
\end{bmatrix}^{-1}
\begin{bmatrix}
\langle W_1, g \rangle \\
\langle W_2, g \rangle \\
\vdots \\
\langle W_N, g \rangle 
\end{bmatrix}.
\]

To use the moment method to solve our new EFIE, the following two steps need to be completed: first, perform the process of dividing up the surfaces into segments in the desired electromagnetics problem, then define some appropriate expansion and weighting functions. These two steps are outlined in the next section.
3.2. Pulse Definitions

At this point all problems are assumed to be axially symmetric about the z-axis; and thin-wire [14] assumptions are used in all regions (i.e. \( -jw\rho_l = \frac{\partial I(z)}{\partial z} \)).

The cylindrical coordinate system illustrated in Appendix A is defined on all the problems presented here. The point matching technique [14] will be used here to solve for the unknown currents and charges in the derived EFIE. The point matching technique uses unit pulses as expansion functions and delta functions as weighting functions. The illustration of dividing a surface is performed here because the domain of the unknowns are related to a physical location. The known expansion functions for the unknown charge will be denoted as \( u_n \) and the known expansion functions for the unknown current will be denoted as \( v_n \). First, each surface in a problem is divided into equal segment lengths. The coordinates representing \( u_n \) are denoted by \( s_n \) and the segment length on the \( n^{th} \) surface is denoted by \( \Delta s_n \). It is possible that two surfaces may not have the same number of segments or have the same segment length. To illustrate this segmentation, an axially symmetric surface about the z-axis is shown in Figure 5. The surface of the conductor is divided into segments and a unit pulse \( (u_n) \) is defined on each segment by

\[
    u_n = \begin{cases} 
    1, & \bar{r} \in s_n \\
    0, & \text{o.w.}
    \end{cases} 
\]  

(3.9)

with a unit vector \( \hat{u}_n \) defined on each pulse in the direction of the surface definition. \( u_n \) will be referred to as a unit pulse and the unknown constant over \( u_n \) is total surface charge density and is denoted by \( \rho_n \). \( \rho_n \) is referred to as a charge pulse. The surface is also divided into a shifted version of the previous segmentation scheme and the unit
pulses \( (v_n) \) are defined as
\[
v_n = \begin{cases} 
1, & \bar{r} \in s_{n+} \\
0, & \text{o.w.} 
\end{cases}
\]  
(3.10)

where \( s_{n+} \) denotes the shifted version of \( s_n \) and the segment length on the surface \( v_n \) is denoted by \( \Delta s_{n+} \). A unit vector \( \hat{e}_n \) is defined on each pulse in the direction of the surface definition. The unknown constant over \( v_n \) is surface current density and is denoted by \( J_n \). \( J_n \) is referred to as a current pulse. This definition is a result of the current continuity equation \( \nabla' \cdot \bar{J}_s = -j\omega \rho_s \). This says that the derivative of the current is equal to \(-j\omega\) times the charge. Since \( v_n \) is shifted from \( u_n \) a difference approximation can be used with the current to solve for charge. Also, the current pulse that intersects the \( z \)-axis is defined to have a magnitude of zero.

The corners with a unit pulse wrapped around them in Figure 5 contain both
unknown polarized and free current. This means that two different pulse functions need to be defined. One unit pulse wraps around the corner to conserve free current on the conductor/dielectric interface and another unit pulse wraps down the edge in the +z-direction to conserve polarized current on the dielectric/dielectric interface.

Under the point matching technique, the integral equations are enforced at discrete points along each surface. The points where the boundary conditions are enforced are now denoted as *match points*. These points are represented with the unprimed vectors. The notations for the source points are left the same and are still represented by the primed vectors. The match points are defined to be in the middle of each $u_n$ for all quasi-static surfaces. The match points on the full-wave surfaces are defined to be in the middle of each $v_n$.

### 3.3. Discrete Version of the Quasi-Static Equations

Now we’ll take a look at the EFIE with match points in the quasi-static region. To solve these integral equations the unknown charge will have to be expanded into known expansion functions and unknown charge pulses. Using (3.9) we get the following expansion for charge:

$$\rho_{tot} = \sum_n u_n \rho_n.$$  (3.11)

The unknown current will also be expanded into known expansion functions and unknown current pulses. This then gives

$$\vec{J}_{s_i} = \sum_n v_n J_n \hat{v}_n$$  (3.12)

where $v_n$ is given in (3.10). Since thin wire assumptions are used in all regions, the
line current $I_n$ is related to the surface current density $J_s$ by

$$I = 2\pi a J_s$$  \hspace{1cm} (3.13)

where $a$ is the radius of the wire. Using (3.13), the unknown surface current density in the previously derived integral equations can be reduced to unknown line current. This then reduces the surface integration to line integration. This then gives the following approximation for the line current:

$$I = \sum_n v_n I_n \hat{v}$$  \hspace{1cm} (3.14)

where $\hat{v}$ is a unit vector in the direction of the source current. The first integral equation considered is (2.73) at point $\bar{p}$ on $S_{12}$ with $i = 1$ and $j = 2$. Using (3.11) and (3.14), we can write

$$
\left(\frac{\varepsilon_r_1 + \varepsilon_r_2}{2\varepsilon_0}\right) \rho_m(\bar{r}_p) = - (\varepsilon_r_1 - \varepsilon_r_2) \vec{E}_{\text{inc}}(\bar{r}_p) - \frac{(\varepsilon_r_1 - \varepsilon_r_2)}{4\pi\varepsilon_0} \hat{n}_1 \cdot \sum_n \int_{s_n} u_n \rho_n \frac{\hat{R}}{R^3} ds'
$$

(3.15)

where

$$\vec{E}_{\text{inc}} = \frac{-1}{4\pi} \left[ \hat{n}_1 \cdot \hat{z} \sum_n \int_{s_n} j\omega\mu_0 v_n I_n \phi_1 dz' - \hat{z}' \frac{1}{j\omega\varepsilon_0} \frac{\partial}{\partial n} \sum_n \frac{I_{n+1} - I_n}{\Delta z'} \int_{s_n} v_n \phi_1 dz' \right].$$

(3.16)

$\frac{\partial}{\partial n}$ is the partial derivative in the normal direction, and the derivatives in (3.16) are evaluated similarly to the methods outlined by Harrington [4]. $\hat{n}_1$ is defined in Figure 6. If the surface is defined from $N_k$ to $N_{k+1}$, then the normal vector is defined to be pointing from $S_{12}$ into region 1. This argument can be generalized to any number of regions. The integrals in (3.15) and (3.16) are only evaluated over the unit pulses
that contain the source points of interest. Notice that the thin wire assumptions simplify the integrals in (3.16) to line integrals w.r.t. $z'$. Similarly, match points on the conductor/dielectric interface, (2.75) can be written as

$$V_k = \psi_{inc}(\vec{r}) + \frac{1}{4\pi\varepsilon_0} \sum_n \int_{s_n} u_n \rho_n \frac{1}{R} ds'$$  \hspace{1cm} (3.17)

where

$$\psi_{inc} = (z - z_0) \frac{1}{4\pi} \frac{1}{z'} \sum_n \int_{s_n+} j\omega\mu_0 v_n I_n \tilde{\phi}_1 dz' - \frac{1}{4\pi j\omega\varepsilon_0} \frac{z'}{z'} \sum_n \frac{I_{n+1} - I_n}{\Delta z'} \int_{s_n} v_n \phi_1 dz'.$$  \hspace{1cm} (3.18)

Using cylindrical coordinates, $R$ can be taken as

$$R = \sqrt{(\rho\cos\phi - \rho'\cos\phi')^2 + (\rho\sin\phi - \rho'\sin\phi')^2 + (z - z')^2}$$  \hspace{1cm} (3.19)

for (3.15) and (3.17). From symmetry, we can take $\phi = \pi/2$. Then $R$ reduces to

$$R = \sqrt{(\rho'\cos\phi')^2 + (\rho - \rho'\sin\phi')^2 + (z - z')^2}.$$  \hspace{1cm} (3.20)
3.4. Discrete Version of the Full-Wave Equations

Next, we’ll take a look at (2.77). A very similar argument for the discrete version of the two integrals in (2.78) is described in detail by Harrington [4] and other numerical aspects can be found in [15]. The unknowns in the EFIE will be expanded using equations (3.11) and (3.14). This gives the discrete version of (2.77) as

\[ \bar{E}_1(\bar{r}) = 0 \]  

(3.21)

where

\[ \bar{E}_1(\bar{r}) = \hat{z} t \frac{1}{4\pi} \sum_n j \omega \mu_0 v_n I_n \phi_1 dz' + \hat{z} t \frac{1}{4\pi \varepsilon_0} \sum_n \frac{I_{n+1} - I_n}{\Delta z'} \int_{s_n} v_n \phi_1 dz' \]

\[ + \bar{E}^Q(\rho, s) \]  

(3.22)

and

\[ \bar{E}^Q(\rho, s) = \hat{z} t \frac{1}{4\pi} \sum_n j \omega \mu_0 v_n I_n \phi_1 dz' - \hat{z} \frac{1}{4\pi \varepsilon_0} \sum_n \int_{s_n} \rho_n \phi_1 ds'. \]  

(3.23)

\( \hat{z} \) is the tangent unit vector at the match point and \( t = \hat{z} \cdot \hat{z}' \). Since the match points in the full-wave region are taken to be on the z-axis \( \Rightarrow r = 0 \), thus reducing (3.20) to

\[ R = \sqrt{(\rho' \cos \phi')^2 + (\rho' \sin \phi')^2 + (z - z')^2}. \]  

(3.24)

3.5. Discrete Version of the Current Continuity Equations

Finally, we’ll look at the discrete version of the current continuity equation. In (2.80) the unknowns are both currents and charges and they are approximated using (3.11) and (3.14). Since we are using thin-wire assumptions in all regions, (2.80) can
be written as

\[ v_{n+k}I_{n+k} \hat{v}_{n+k} - v_n I_n \hat{v}_n = -j\omega \varepsilon_{r_1} \sum_n u_n \rho_n A_n \]  

(3.25)

for free charge and

\[ v_{n+k}I_{n+k} \hat{v}_{n+k} - v_n I_n \hat{v}_n = -j\omega \varepsilon_0 (\varepsilon_{r_i} - 1) \sum_n u_n \rho_n A_n \]  

(3.26)

for the polarized charges where \( A_n \) is the area of the segment the charge pulse is defined on. The current continuity will have to applied twice at the junction in Figure 7. This will ensure the continuity of the free charge and the polarized charge is enforced. Now we have enough discrete integral equations to be evaluated by the MOM.

![Figure 7. Continuity of the Free and Polarized Charge.](image)

### 3.6. Hybrid Final Impedance Matrix

The following matrix is a summary of the discrete hybrid EFIE. It is divided into four main components. The first component consists of the quasi-static equations. This includes the EFIE for match points on the dielectric/dielectric and conductor/dielectric interfaces. The second component consists of the full-wave equations, the third component consists of the current continuity equations, and the final
component consists of any additional current continuity equations to solve for the unknown voltages. \( \rho_n \) is the unknown charge in the quasi-static region, \( I_{qn} \) is the unknown current in the quasi-static region, \( I_n \) is the unknown current in the full-wave region, and \( V_k \) is the unknown voltage on the \( k^{th} \) conductor.

\[
\begin{bmatrix}
\text{Quasi – Static Equations} \\
\text{Full – Wave Equations} \\
\text{Current Continuity Equations} \\
\text{Additional Continuity Equations for Unknown Voltages}
\end{bmatrix}
\begin{bmatrix}
\rho_1 \\
I_1 \\
I_{q1} \\
V_1 \\
\vdots
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
E_{inc} \\
\vdots
\end{bmatrix}
\]

We can then solve for the unknown charges, currents, and voltages by evaluating the following matrix. The inverse of the matrix was evaluated by using the \textit{inv} [16] command in Matlab.

\[
\begin{bmatrix}
\rho_1 \\
I_1 \\
I_{q1} \\
V_1 \\
\vdots
\end{bmatrix}
= \begin{bmatrix}
\text{Quasi – Static Equations} \\
\text{Full – Wave Equations} \\
\text{Current Continuity Equations} \\
\text{Additional Continuity Equations for Unknown Voltages}
\end{bmatrix}^{-1}
\begin{bmatrix}
0 \\
E_{inc} \\
\vdots
\end{bmatrix}
\]
3.7. Evaluating the Derivatives in the Hybrid EFIE

The central difference approximation is used to approximate the derivatives in all EFIE. The boundary condition on the dielectric/dielectric interface enforces the continuity of the normal components of the electric flux density. First, consider the $\vec{E}_{inc}$ term in (3.15). In particular, the second integral in (3.16) has two derivatives to be evaluated, $\frac{\partial}{\partial n}$ and $\frac{\partial}{\partial z'}$. To illustrate both derivatives, consider the points $P_{m+1/2}$ and $P_{m-1/2}$ in Figure 8. $P_{m+1/2}$ is defined to be at 1/20th of the surface segment away from $P_m$ in the $+\hat{n}$ direction, and $P_{m-1/2}$ is defined to be at 1/20th of the surface segment away from $P_m$ in the $-\hat{n}$ direction. This means that $\frac{\partial}{\partial n}$ is taken in the direction normal to the dielectric/dielectric interfaces.

Define $\vec{R}_{m+1/2}$ to be the vector from the unit pulse $u_{k+1}$ to $P_{m+1/2}$, and let $\vec{R}_{m-1/2}$ be the vector from the unit pulse $u_{k+1}$ to $P_{m-1/2}$. $\rho_{k+1}$ is the unknown charge pulse over $u_{k+1}$, thus $\vec{R}_{m+1/2}$ and $\vec{R}_{m-1/2}$ are the vectors from the charge contribution in the full-wave region to $P_{m+1/2}$ and $P_{m-1/2}$, respectfully. Similarly, define $\vec{R}_{m+1/2}$ to be the vector from the unit pulse $u_k$ to $P_{m+1/2}$ and let $\vec{R}_{m-1/2}$ be the vector from the unit pulse $u_k$ to $P_{m-1/2}$. Note that the unknowns, $\rho_{k+1}$ and $\rho_k$, are centered about the edges of the $k^{th}$ unknown current pulse. After many steps, the discrete version of the second integral in (3.16) can be written in the following generalized manner [4]

$$-\frac{z'}{4\pi j\omega\varepsilon_0} \frac{D_{m+1/2} - D_{m-1/2}}{\Delta n}$$  \hspace{1cm} (3.27)

where

$$D_{m+1/2} = \sum_n \int_{s_n} v_n I_n \frac{\phi_1(\vec{R}_{m+1/2}^{n+1}) - \phi_1(\vec{R}_{m+1/2}^n)}{\Delta z'} dz'$$  \hspace{1cm} (3.28)

and

$$D_{m-1/2} = \sum_n \int_{s_n} v_n I_n \frac{\phi_1(\vec{R}_{m-1/2}^{n+1}) - \phi_1(\vec{R}_{m-1/2}^n)}{\Delta z'} dz'.$$  \hspace{1cm} (3.29)

Notice that (3.27) is now written in terms of one unknown, $I_n$. $D_{m+1/2}$ is the derivative
at $P_{m+1/2}$ w.r.t. the source points $\rho_{n+1}$ and $\rho_n$, and $\Delta z'$ is the full-wave current segment length. Similarly, $D_{m-1/2}$ is the derivative at $P_{m-1/2}$ w.r.t. the source points $\rho_{n+1}$ and $\rho_n$. (3.27) is the normal derivative about $P_m$ while enforcing the boundary conditions at the match point where $\Delta n = |P_{m+1/2} - P_{m-1/2}|$. This same procedure can now be applied to each match point on the dielectric/dielectric interfaces in the quasi-static region. $P_{m\pm 1/2}$ can also be defined to take the tangential derivative in the full-wave region. $P_{m+1/2}$ and $P_{m-1/2}$ are taken to be a half of a segment away from $P_m$ in the positive and negative tangential directions, respectively. This is shown in Figure 9. Special care should be taken when evaluating (3.22) for match points.
$P_m$ on the interface between the full-wave regions and quasi-static regions. $\Delta z$ is still defined as the distance between $P_{m+1/2}$ and $P_{m-1/2}$ but $|P_{m+1/2} - P_m| \neq |P_m - P_{m-1/2}|$. This can easily be overlooked when writing code to evaluate these derivatives.

Special care should also be taken if the match points are in the full-wave region and the source points are on the boundary between the full-wave region and the quasi-static region. In particular, using (3.28) and (3.29) we can write the second term in (3.22) as

$$\frac{\partial}{\partial z'} \sum_n \int_{s_n} \frac{\partial}{\partial z'} v_n I_n \phi_1 dz' \approx \frac{D_{m+1/2} - D_{m-1/2}}{\Delta z}$$

$$= \frac{1}{\Delta z} \sum_n v_n I_n \frac{\phi_1(\bar{R}_{m+1/2}^{n+1}) - \phi_1(\bar{R}_{m+1/2}^n)}{\Delta z'}$$

$$- \frac{1}{\Delta z} \sum_n v_n I_n \frac{\phi_1(\bar{R}_{m-1/2}^{n+1}) - \phi_1(\bar{R}_{m-1/2}^n)}{\Delta z'}.$$  (3.30)

Figure 9. Conductor/Dielectric Tangential Derivative.
If $u_{k+1}$ is the unit pulse that transitions into the quasi-static region, then

$$\frac{\phi_1(\bar{R}^{k+1}_{m+1/2}) - \phi_1(\bar{R}^{k+1}_{m-1/2})}{\Delta z} \quad (3.31)$$

is the contribution at the match point as a result of the quasi-static charge defined over $u_{k+1}$. This contribution is already calculated by the second term in (3.23) and is approximated by

$$\frac{\partial}{\partial z} \sum_n \int_{s_n} u_n \rho_n \phi_i \, ds' \approx \sum_n \int_{s_n} u_n \rho_n \frac{\phi_i(\bar{R}^{n+1}_{m+1/2}) - \phi_i(\bar{R}^{n+1}_{m-1/2})}{\Delta z} \quad (3.32)$$

To avoid this calculation from happening twice, (3.31) is forced to be zero for the calculations in the MOM.

3.8. Evaluating the Integrals in the Hybrid EFIE

The integrals in the hybrid EFIE need to be evaluated numerically. The line integrals are defined in Figure 10. To perform line integrals the $quad$ [16] command in Matlab was chosen. $quad$ uses an adaptive Simpson quadrature to integrate...
over specified limits. To perform surface integrals the `dbquad` [16] routine calls the routine `quad` twice to evaluate the double integral over specified limits. The line integrals on the vertical surfaces were always calculated in the \( +z \)-direction regardless of the direction the surface was defined. The line integrals on the horizontal surfaces are always calculated in the \( +\rho \)-direction regardless of the direction the surface was defined.

Special care should be taken when the quasi-static match points and source points are defined on the same segment. To avoid a singularity while evaluating (3.15) the surface was divided into an odd number of segments. This prevented the possibility of defining the source and match points at the same location.

### 3.9. Sources

The expression for the tangential components of the electric field on the conductor/dielectric interface can be written as \( \vec{E}_{\text{tan}} + \vec{E}_{\text{tan}} = 0 \) where \( \vec{E}_{\text{tan}} \) is the tangential component of the scattered field and \( \vec{E}_{\text{tan}} \) is the tangential component of the incident field. The derived EFIE are the expressions representing the scattered field as a result of an incident wave. Now we need to write a few expressions to represent an incident wave that will excite the problem.

#### 3.9.1. Delta Source

The first source considered is the Delta Source [14]. This type of source is probably the most common source implemented in the MOM. This type of source generates an electric field by applying a voltage source, \( V_A \), across one of the full-wave segments. The voltage points can be seen in Figure 11. The electric potential between points \( a \) and \( b \) can be written as

\[
V_b - V_a = - \int_a^b \vec{E}_{\text{tan}} \cdot d\vec{l}.
\]  

(3.33)

Assuming \( \vec{E}_{\text{tan}} \) is constant, we can integrate (3.33) across the \( \delta \)-gap in Figure 11.
giving \( V_A = E_i \delta \) which then gives

\[
E_i = \frac{V_A}{\delta}.
\] (3.34)

It is assumed that the \( \delta \)-gap is small enough such that no fringing of the electric field exists. (3.34) has two known values, \( V_A \) and \( \delta \), and can be implemented in the MOM to satisfy the expression for \( \vec{E}^s_{\tan} + \vec{E}^i_{\tan} = 0 \) on the conductor/dielectric interface. Delta sources are a good way to simulate the fields from an antenna when it is driven at a single point with a voltage source. From the calculated currents the input impedance \( Z_{in} \) at the \( k^{th} \) driving point can then be calculated by

\[
Z_{in} = \frac{V_A}{I_k}.
\] (3.35)

3.10. Hybrid Validation Results

A simple problem was chosen to implement the integral equations derived for the hybrid method. The problem in Figure 12 is a dipole in free space with a quasi-static region located at one end. A conductor/dielectric interface transitions between the full-wave and quasi-static regions and a dielectric region with \( \varepsilon_r = 3\varepsilon_0 \) terminates the antenna. The entire problem is surrounded by air (\( \varepsilon_r = \varepsilon_0 \)). Matlab was chosen to solve the integral equations of the hybrid method. Some existing Fortran code [17] was located and compiled as a comparison to the Matlab code. The full-wave region
Figure 12. Hybrid Validation Example.

was divided into three segments, and the quasi-static region was divided into seven segments. Again, a simple problem was chosen because it was easier to manage the large amount of data generated by a given problem. Table 1 shows a comparison between the charges and currents the Matlab code and the Fortran code calculated using the original hybrid method. The input impedance calculated by Matlab was

\[ Z_{in,\text{Matlab}} = 39.651 - 236.652i\Omega, \]

and the input impedance calculated by the Fortran code was

\[ Z_{in,\text{Fortran}} = 39.7832 - 234.316i\Omega, \]
Table 1. Comparison Between Matlab and Fortran Calculations.

<table>
<thead>
<tr>
<th>$\rho$</th>
<th>coord.(m)</th>
<th>$z$ coord.(m)</th>
<th>Matlab</th>
<th>Hybrid</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho_1$</td>
<td>0.750e-04</td>
<td>0.557e-01</td>
<td>3.39e-08-9.04e-09i</td>
<td>0.33e-07-0.90e-08i</td>
</tr>
<tr>
<td>$\rho_2$</td>
<td>0.562e-04</td>
<td>0.568e-01</td>
<td>1.37e-07-3.50e-08i</td>
<td>0.14e-06-0.38e-07i</td>
</tr>
<tr>
<td>$\rho_3$</td>
<td>0.187e-04</td>
<td>0.568e-01</td>
<td>6.85e-08-1.74e-08i</td>
<td>0.70e-07-0.19e-07i</td>
</tr>
<tr>
<td>$\rho_4$</td>
<td>0.750e-04</td>
<td>0.578e-01</td>
<td>1.01e-10-2.64e-11i</td>
<td>0.94e-10-0.25e-10i</td>
</tr>
<tr>
<td>$\rho_5$</td>
<td>0.750e-04</td>
<td>0.600e-01</td>
<td>9.42e-12-2.40e-12i</td>
<td>0.88e-11-0.22e-11i</td>
</tr>
<tr>
<td>$\rho_6$</td>
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<td>0.611e-01</td>
<td>2.45e-10-9.41e-11i</td>
<td>0.24e-09-0.94e-10i</td>
</tr>
<tr>
<td>$\rho_7$</td>
<td>0.187e-04</td>
<td>0.611e-01</td>
<td>2.44e-10-9.39e-11i</td>
<td>0.24e-09-0.94e-10i</td>
</tr>
<tr>
<td>$I_1$</td>
<td>0.750e-04</td>
<td>0.182e-01</td>
<td>6.88e-04+4.11e-03i</td>
<td>0.70e-03+0.41e-02i</td>
</tr>
<tr>
<td>$I_2$</td>
<td>0.750e-04</td>
<td>0.364e-01</td>
<td>7.38e-04+3.22e-03i</td>
<td>0.75e-03+0.32e-02i</td>
</tr>
<tr>
<td>$I_3$</td>
<td>0.750e-04</td>
<td>0.547e-01</td>
<td>1.34e-04+5.08e-04i</td>
<td>0.13e-03+0.50e-03i</td>
</tr>
<tr>
<td>$I_{q1}$</td>
<td>0.750e-04</td>
<td>0.568e-01</td>
<td>2.04e-05+8.01e-05i</td>
<td>0.22e-04+0.82e-04i</td>
</tr>
<tr>
<td>$I_{q2}$</td>
<td>0.375e-04</td>
<td>0.568e-01</td>
<td>9.71e-07+3.80e-06i</td>
<td>0.10e-05+0.39e-05i</td>
</tr>
<tr>
<td>$I_{q3}$</td>
<td>0.750e-04</td>
<td>0.568e-01</td>
<td>1.36e-05+5.34e-05i</td>
<td>0.14e-04+0.55e-04i</td>
</tr>
<tr>
<td>$I_{q4}$</td>
<td>0.750e-04</td>
<td>0.589e-01</td>
<td>1.32e-05+5.21e-05i</td>
<td>0.14e-04+0.53e-04i</td>
</tr>
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<td>$I_{q5}$</td>
<td>0.750e-04</td>
<td>0.611e-01</td>
<td>1.32e-05+5.20e-05i</td>
<td>0.14e-04+0.53e-04i</td>
</tr>
<tr>
<td>$I_{q6}$</td>
<td>0.375e-04</td>
<td>0.611e-01</td>
<td>1.32e-05+5.20e-05i</td>
<td>0.14e-04+0.53e-04i</td>
</tr>
<tr>
<td>$V_1$</td>
<td>1.30e+00</td>
<td>3.70e-01</td>
<td>1.2e+01-0.37e+00i</td>
<td>0.12e+01-0.37e+00i</td>
</tr>
</tbody>
</table>

thus showing successful calculations by Matlab. The results in Table 1 helped tremendously to establish much needed information pertaining to the derivations, derivatives, and numerical integration in the hybrid program. The Matlab code that generated Table 1 is located in Appendix B.
CHAPTER 4. DERIVATION OF THE NEW EFIE

4.1. Introduction

After successfully implementing the hybrid method new EFIE will be derived and implemented. The work presented in Chapters 2 and 3 provides a good foundation and guide for the derivation of the new equations. The format of the next two chapters will be very similar to the previous two, except the terms including the inductive effects will be present. Where appropriate, the differences between these two chapters will be pointed out.

4.2. Description of General Electromagnetics Problem

For illustration the surfaces in Figure 1 are presented in Figure 13. Again, the

![Figure 13. General Problem for the New EFIE Derivations.](image-url)
electric field in the $i^{th}$ region can be written as [2]

$$
\vec{E}_i(\vec{r}) = \begin{cases} 
T \oint_{\partial_i} \vec{Q}_i \, ds' & \text{\(\vec{r}\) in i} \\
0 & \text{\(\vec{r}\) not in i}
\end{cases}
$$  \hspace{1cm} (4.1)

where

$$
\vec{Q}_i = \frac{-1}{4\pi} \left[ j\omega \mu_i (\hat{n}'_i \times \vec{H}_i) \phi_i - (\hat{n}'_i \times \vec{E}_i) \times \nabla' \phi_i - (\hat{n}'_i \cdot \vec{E}_i) \nabla' \phi_i \right],
$$  \hspace{1cm} (4.2)

and $k_i = \omega \sqrt{\mu_i \varepsilon_i}$. The fields in (4.1) are assumed to vary with $e^{j\omega t}$ and $\partial_i$ indicates that the integral is evaluated over all the smooth surfaces bounding the $i^{th}$ region of interest (note that the integral is a principle value integral [7]-[8]). The following derivations do not assume any axial symmetry in any region but quasi-static assumptions are still present. Notice that we have a relative permeability other than unity in (4.2). The same arguments for the equivalence principle in Chapter 2 with $\mu_0$ hold for $\mu_i$.

4.3. Quasi-Static Equations

The quasi-static assumptions presented in section (2.3) are still valid here. A quick outline of these assumptions are shown here. If $k_i D \ll 1$, where $D$ is the overall maximum dimension of the quasi-static region, then

$$
\phi_i \approx \frac{1}{R},
$$  \hspace{1cm} (4.4)

$$
\nabla \phi_i \approx \frac{-\hat{R}}{R^3}
$$  \hspace{1cm} (4.5)

and

$$
\nabla' \phi_i = -\nabla \phi_i.
$$  \hspace{1cm} (4.6)
With these approximations (4.2) can be written as

\[
\bar{Q}_{i,\text{quasi}} = -\frac{1}{4\pi} \left[ j\omega \mu_i (\hat{n}'_i \times \bar{H}_i) \frac{1}{R} - (\hat{n}'_i \times \bar{E}_i) \times \frac{\bar{R}}{R^3} - (\hat{n}'_i \cdot \bar{E}_i) \frac{\bar{R}}{R^3} \right]. \tag{4.7}
\]

For source points on the conductor/dielectric interface, \( \hat{n}'_i \times \bar{H}_i \) represents the magnetic induction effects in the quasi-static region. For these derivations it is assumed that the electrostatic effects are not dominant \( \Rightarrow \hat{n}'_i \times \bar{H}_i \) cannot be neglected.

### 4.3.1. General Expression for the Electric Field

To evaluate the new EFIE a general expression for the electric field needs to be determined. Consider regions 1 and 2 in Figure 13. Using (4.1), the electric field in region 1 can be written as

\[
\vec{E}_1(\vec{r}) = 1 \int_{S_{13}} \bar{Q}_1 ds' + 1 \int_{S_{12}} \bar{Q}_{1,\text{quasi}} ds' + 1 \int_{S_{14}} \bar{Q}_{1,\text{quasi}} ds'. \tag{4.8}
\]

Similarly, the electric field in region 2 can be written as

\[
\vec{E}_2(\vec{r}) = 1 \int_{S_{12}} \bar{Q}_{2,\text{quasi}} ds' + 1 \int_{S_{24}} \bar{Q}_{2,\text{quasi}} ds'. \tag{4.9}
\]

This then gives

\[
\vec{E}_1(\vec{r}) = 1 \int_{S_{13}} \bar{Q}_1 ds' + 1 \int_{S_{12}} \bar{Q}_{1,\text{quasi}} ds' + 1 \int_{S_{14}} \bar{Q}_{1,\text{quasi}} ds' \\
= \frac{1}{4\pi} \int_{S_{13}} -1 \left[ j\omega \mu_1 (\hat{n}'_1 \times \bar{H}_1) \phi_1 - (\hat{n}'_1 \times \bar{E}_1) \times \nabla' \phi_1 - (\hat{n}'_1 \cdot \bar{E}_1) \nabla' \phi_1 \right] ds' + \\
1 \int_{S_{12}} -1 \left[ j\omega \mu_1 (\hat{n}'_1 \times \bar{H}_1) \frac{1}{R} - (\hat{n}'_1 \times \bar{E}_1) \times \frac{\bar{R}}{R^3} - (\hat{n}'_1 \cdot \bar{E}_1) \frac{\bar{R}}{R^3} \right] ds' + \\
1 \int_{S_{14}} -1 \left[ j\omega \mu_1 (\hat{n}'_1 \times \bar{H}_1) \frac{1}{R} - (\hat{n}'_1 \times \bar{E}_1) \times \frac{\bar{R}}{R^3} - (\hat{n}'_1 \cdot \bar{E}_1) \frac{\bar{R}}{R^3} \right] ds'. \tag{4.10}
\]
and

\[ E_2(\vec{r}) = \int_{S_{12}} \vec{Q}_{2,\text{quasi}} \, ds' + \int_{S_{24}} \vec{Q}_{2,\text{quasi}} \, ds' \]

\[ = \int_{S_{12}} \frac{-1}{4\pi} \left[ j\omega \mu_2 (\hat{n}'_2 \times \vec{H}_2) \frac{1}{R} - (\hat{n}'_2 \times \vec{E}_2) \times \frac{\vec{R}}{R^3} - (\hat{n}'_2 \cdot \vec{E}_2) \frac{\vec{R}}{R^3} \right] \, ds' + \]

\[ = \int_{S_{24}} \frac{-1}{4\pi} \left[ j\omega \mu_2 (\hat{n}'_2 \times \vec{H}_2) \frac{1}{R} - (\hat{n}'_2 \times \vec{E}_2) \times \frac{\vec{R}}{R^3} - (\hat{n}'_2 \cdot \vec{E}_2) \frac{\vec{R}}{R^3} \right] \, ds' \quad (4.11) \]

Now, the expression for the electric field, with \( \vec{r} \) in the \( i^{th} \) region, can be written as

\[ E_i(\vec{r}) = E_1(\vec{r}) + E_2(\vec{r}) \]

\[ = 1 \vec{E}_{inc} + \]

\[ = \int_{S_{12}} \frac{-1}{4\pi} \left[ j\omega \mu_1 (\hat{n}'_1 \times \vec{H}_1) \frac{1}{R} - (\hat{n}'_1 \times \vec{E}_1) \times \frac{\vec{R}}{R^3} - (\hat{n}'_1 \cdot \vec{E}_1) \frac{\vec{R}}{R^3} \right] \, ds' + \]

\[ = \int_{S_{12}} \frac{-1}{4\pi} \left[ j\omega \mu_2 (\hat{n}'_2 \times \vec{H}_2) \frac{1}{R} - (\hat{n}'_2 \times \vec{E}_2) \times \frac{\vec{R}}{R^3} - (\hat{n}'_2 \cdot \vec{E}_2) \frac{\vec{R}}{R^3} \right] \, ds' + \]

\[ = \int_{S_{14}} \frac{-1}{4\pi} \left[ j\omega \mu_1 (\hat{n}'_1 \times \vec{H}_1) \frac{1}{R} - (\hat{n}'_1 \times \vec{E}_1) \times \frac{\vec{R}}{R^3} - (\hat{n}'_1 \cdot \vec{E}_1) \frac{\vec{R}}{R^3} \right] \, ds' + \]

\[ = \int_{S_{24}} \frac{-1}{4\pi} \left[ j\omega \mu_2 (\hat{n}'_2 \times \vec{H}_2) \frac{1}{R} - (\hat{n}'_2 \times \vec{E}_2) \times \frac{\vec{R}}{R^3} - (\hat{n}'_2 \cdot \vec{E}_2) \frac{\vec{R}}{R^3} \right] \, ds' \quad (4.12) \]

where

\[ \vec{E}_{inc} = \int_{S_{13}} \frac{-1}{4\pi} \left[ j\omega \mu_1 (\hat{n}'_1 \times \vec{H}_1) \phi_1 - (\hat{n}'_1 \times \vec{E}_1) \times \nabla' \phi_1 - (\hat{n}'_1 \cdot \vec{E}_1) \nabla' \phi_1 \right] \, ds' \quad (4.13) \]

is the contribution from \( S_{13} \). Notice that the approximation in (4.4) is not used in (4.13). This is because the source points are in the full-wave region and not in the same quasi-static region as the field points. Since \( \hat{n}'_1 = -\hat{n}'_2 \), we can group the terms
in (4.12) as

\[
\vec{E}_i(\vec{r}) = 1 \left[ \vec{E}_{inc} + \frac{-1}{4\pi} \int_{S_{14}} \left[ j\omega \mu_1 \left( \hat{n}'_1 \times \vec{H}_1 \right) \frac{1}{R} - \left( \hat{n}'_1 \cdot \vec{E}_1 \right) \frac{\vec{R}}{R^3} \right] ds' + \frac{-1}{4\pi} \int_{S_{24}} \left[ j\omega \mu_2 \left( \hat{n}'_2 \times \vec{H}_2 \right) \frac{1}{R} - \left( \hat{n}'_2 \cdot \vec{E}_2 \right) \frac{\vec{R}}{R^3} \right] ds' + \frac{-1}{4\pi} \int_{S_{12}} \left[ j\omega \left( \mu_1 \vec{H}_1 - \mu_2 \vec{H}_2 \right) \frac{1}{R} - \hat{n}'_1 \times (\vec{E}_1 - \vec{E}_2) \times \frac{\vec{R}}{R^3} - \hat{n}'_2 \cdot (\vec{E}_1 - \vec{E}_2) \frac{\vec{R}}{R^3} \right] ds' \right]. \tag{4.14}
\]

Since the boundaries on \( S_{13}, S_{14}, \) and \( S_{24} \) are composed of a perfect conductor and dielectric, we have

\[
\hat{n}'_1 \times \vec{E}_1 = 0 \tag{4.15}
\]

and

\[
\hat{n}'_2 \times \vec{E}_2 = 0, \tag{4.16}
\]

giving

\[
\vec{E}_i(\vec{r}) = 1 \left[ \vec{E}_{inc} + \frac{-1}{4\pi} \int_{S_{14}} \left[ j\omega \mu_1 \left( \hat{n}'_1 \times \vec{H}_1 \right) \frac{1}{R} - \left( \hat{n}'_1 \cdot \vec{E}_1 \right) \frac{\vec{R}}{R^3} \right] ds' + \frac{-1}{4\pi} \int_{S_{24}} \left[ j\omega \mu_2 \left( \hat{n}'_2 \times \vec{H}_2 \right) \frac{1}{R} - \left( \hat{n}'_2 \cdot \vec{E}_2 \right) \frac{\vec{R}}{R^3} \right] ds' + \frac{-1}{4\pi} \int_{S_{12}} \left[ j\omega \left( \mu_1 \vec{H}_1 - \mu_2 \vec{H}_2 \right) \frac{1}{R} - \hat{n}'_1 \times (\vec{E}_1 - \vec{E}_2) \times \frac{\vec{R}}{R^3} - \hat{n}'_2 \cdot (\vec{E}_1 - \vec{E}_2) \frac{\vec{R}}{R^3} \right] ds' \right]. \tag{4.17}
\]
Also, since $S_{12}$ is made up of two perfect dielectrics we have

\[ \hat{n}'_1 \times (\vec{H}_1 - \vec{H}_2) = 0, \quad (4.18) \]

\[ \hat{n}'_1 \times (\vec{E}_1 - \vec{E}_2) = 0, \quad (4.19) \]

and

\[ \hat{n}'_1 \cdot (\vec{D}_1 - \vec{D}_2) = \hat{n}'_1 \cdot \varepsilon_0 (\varepsilon_{r_1} \vec{E}_1 - \varepsilon_{r_2} \vec{E}_2) = 0. \quad (4.20) \]

Using (4.20) we can write a relation for the electric field as

\[ \hat{n}'_1 \cdot \vec{E}_2 = \frac{\varepsilon_{r_1}}{\varepsilon_{r_2}} \hat{n}'_1 \cdot \vec{E}_1 \quad (4.21) \]

and from (4.18) we can write

\[ \hat{n}'_1 \times \vec{H}_1 = \hat{n}'_1 \times \vec{H}_2. \quad (4.22) \]

Using (4.21) on $S_{12}$ we can write

\[ -\hat{n}'_1 \cdot (\vec{E}_1 - \vec{E}_2) = -\hat{n}'_1 \cdot (\vec{E}_1 - \frac{\varepsilon_{r_1}}{\varepsilon_{r_2}} \vec{E}_1) \]
\[ = -\hat{n}'_1 \cdot \vec{E}_1 \left( \frac{\varepsilon_{r_2} - \varepsilon_{r_1}}{\varepsilon_{r_2}} \right) \quad (4.23) \]

and using (4.22) we can also write

\[ j \omega \hat{n}'_1 \times (\mu_1 \vec{H}_1 - \mu_2 \vec{H}_2) = j \omega \mu_1 (\hat{n}'_1 \times \vec{H}_1) - j \omega \mu_2 (\hat{n}'_1 \times \vec{H}_2) \]
\[ = j \omega \mu_1 (\hat{n}'_1 \times \vec{H}_1) - j \omega \mu_2 (\hat{n}'_1 \times \vec{H}_1) \]
\[ = j \omega (\mu_1 - \mu_2) (\hat{n}'_1 \times \vec{H}_1). \quad (4.24) \]
Substituting (4.23) and (4.24) into (4.17) we get

$$\bar{E}_i(\bar{r}) = \frac{1}{2} \left[ \bar{E}_{inc} + \frac{-1}{4\pi} \int_{S_{14}} \left[ j\omega \mu_1 (\hat{n}'_1 \times \bar{H}_1) \frac{1}{R} - (\hat{n}'_1 \cdot \bar{E}_1) \frac{\bar{R}}{R^3} \right] ds' + \frac{-1}{4\pi} \int_{S_{24}} \left[ j\omega \mu_2 (\hat{n}'_2 \times \bar{H}_2) \frac{1}{R} - (\hat{n}'_2 \cdot \bar{E}_2) \frac{\bar{R}}{R^3} \right] ds' + \frac{-1}{4\pi} \int_{S_{12}} \left[ j\omega (\mu_1 - \mu_2) (\hat{n}'_1 \times \bar{H}_1) \frac{1}{R} - \frac{\varepsilon_{r_2} - \varepsilon_{r_1}}{\varepsilon_{r_2}} (\hat{n}'_1 \cdot \bar{E}_1) \frac{\bar{R}}{R^3} \right] ds' \right] \quad (4.25)$$

where, again,

$$\bar{E}_{inc} = \int_{S_{13}} \frac{-1}{4\pi} \left[ j\omega \mu_1 (\hat{n}'_1 \times \bar{H}_1) \phi_1 - (\hat{n}'_1 \times \bar{E}_1) \times \nabla' \phi_1 - (\hat{n}'_1 \cdot \bar{E}_1) \nabla' \phi_1 \right] ds'. \quad (4.26)$$

Notice in (4.25) that the expressions that represent the magnetic inductive effects, $j\omega \mu_1 (\hat{n}'_1 \times \bar{H}_1) \frac{1}{R}$, $j\omega \mu_2 (\hat{n}'_2 \times \bar{H}_2) \frac{1}{R}$, and $j\omega (\mu_1 - \mu_2) (\hat{n}'_1 \times \bar{H}_1) \frac{1}{R}$, are not present in (2.25).

4.3.2. Dielectric/Dielectric Interfaces

Using the expression for the electric field in (4.25), the boundary condition for free charge will be enforced at point $\bar{p}$ on $S_{12}$ shown in Figure 14. Using the definition of the unit normal, described in Figure 14, the normal components of the electric field are related by

$$\hat{n}_1 \cdot (\bar{D}_1 - \bar{D}_2) = \rho_{free} \quad (4.27)$$

with $\bar{D}_i = \varepsilon_i \bar{E}_i$ and $\rho_{free}$ denoting the free surface charge density. Since the boundary is made up of perfect dielectrics, $\rho_{free} = 0$. This then gives

$$\hat{n}_1 \cdot (\varepsilon_{r_1} \bar{E}_1 - \varepsilon_{r_2} \bar{E}_2) = 0. \quad (4.28)$$
As before, to evaluate this boundary condition at $\vec{p}$, the limit \[ \lim_{\vec{r}_1 \to \vec{r}_p} \varepsilon_{r_1} \hat{n}_1 \cdot \vec{E}_1(\vec{r}_1) \]
will have to be evaluated with $\vec{r}_1 \to \vec{r}_p$ from region 1 along the dotted normal vector in Figure 14. This then gives

\[
\lim_{\vec{r}_1 \to \vec{r}_p} \varepsilon_{r_1} \hat{n}_1 \cdot \vec{E}_1(\vec{r}_1) = 2\varepsilon_{r_1} \left[ \hat{n}_1 \cdot \vec{E}_{inc}(\vec{r}_p) + \int_{S_{14}} \left( \frac{1}{R_1} \hat{n}_1' \times \vec{H}_1 \right) \frac{1}{R_1} \frac{\vec{R}_1}{R_1} \, ds' + \int_{S_{24}} \left( \frac{1}{R_2} \hat{n}_2' \times \vec{H}_2 \right) \frac{1}{R_2} \frac{\vec{R}_2}{R_2} \, ds' + \right. \\
\left. \int_{S_{12}} \left( \frac{1}{R} \hat{n}_1' \cdot \vec{E}_1 \right) \frac{\vec{R}}{R^3} \, ds' \right] \quad \text{(4.29)}
\]
Again, it can be shown that [10]-[11]

\[
\lim_{\hat{r}_1 \to \hat{r}_p} \hat{n}_1 \cdot \int_{S_{12}} (\hat{n}'_1 \cdot \hat{E}_1) \frac{\bar{R}}{R^3} ds' = 2\pi (\hat{n}_1 \cdot \hat{E}_1(\bar{r}_p)) + \hat{n}_1 \cdot \int_{S_{12}} (\hat{n}'_1 \cdot \hat{E}_1) \frac{\bar{R}_p}{R_p^3} ds' \quad (4.30)
\]

where the term \(2\pi (\hat{n}_1 \cdot \hat{E}_1)\) is due to the discontinuity of the normal components of the electric field across the boundary. By Stratton [18], \(\hat{n}'_1 \times \hat{H}_1\) is continuous across \(S_{12}\). Thus we can write

\[
\lim_{\hat{r}_1 \to \hat{r}_p} \hat{n}_1 \cdot \int_{S_{12}} (\hat{n}'_1 \times \hat{H}_1) \frac{1}{R} ds' = \hat{n}_1 \cdot \int_{S_{12}} (\hat{n}'_1 \times \hat{H}_1) \frac{1}{R_p} ds'. \quad (4.31)
\]

This then gives

\[
\begin{align*}
\lim_{\hat{r}_1 \to \hat{r}_p} \varepsilon_{r_1} \hat{n}_1 \cdot \hat{E}_1(\bar{r}_1) &= 2\varepsilon_{r_1} \left[ \hat{n}_1 \cdot \hat{E}_{inc}(\bar{r}_p) + \right. \\
& \quad \hat{n}_1 \cdot \frac{1}{4\pi} \int_{S_{14}} \left[ j \omega \mu_1 (\hat{n}'_1 \times \hat{H}_1) \frac{1}{R_1} - (\hat{n}'_1 \cdot \hat{E}_1) \frac{\bar{R}_1}{R_1^3} \right] ds' + \\
& \quad \hat{n}_1 \cdot \frac{1}{4\pi} \int_{S_{24}} \left[ j \omega \mu_2 (\hat{n}'_2 \times \hat{H}_2) \frac{1}{R_2} - (\hat{n}'_2 \cdot \hat{E}_2) \frac{\bar{R}_2}{R_2^3} \right] ds' + \\
& \quad \left. \frac{-1}{4\pi} \left( \frac{\varepsilon_{r_1} - \varepsilon_{r_2}}{\varepsilon_{r_2}} \right) \left( 2\pi (\hat{n}_1 \cdot \hat{E}_1(\bar{r}_p)) + \hat{n}_1 \cdot \int_{S_{12}} (\hat{n}'_1 \cdot \hat{E}_1) \frac{\bar{R}_p}{R_p^3} ds' \right) + \\
& \quad \frac{-1}{4\pi} j \omega (\mu_1 - \mu_2) \hat{n}_1 \cdot \int_{S_{12}} (\hat{n}'_1 \times \hat{H}_1) \frac{1}{R_p} ds' \right] \\
& = \varepsilon_{r_1} \hat{n}_1 \cdot \hat{E}_1(\bar{r}_p) \quad (4.32)
\end{align*}
\]

where \(\bar{R}_1 = \bar{r}_p - \bar{r}_{14}, \bar{R}_2 = \bar{r}_p - \bar{r}_{24}\), and \(\bar{R}_p = \bar{r}_p - \bar{r}_{12}, \bar{r}_{14}, \bar{r}_{24}, \) and \(\bar{r}_{12}\) are vectors indicating the source locations on \(S_{14}, S_{24}\), and \(S_{12}\), respectively. Similarly, to evaluate
this boundary condition at \( \vec{p} \), the limit [10]

\[
\lim_{\vec{r}_2 \to \vec{r}_p} \varepsilon_{r_2} \hat{n}_1 \cdot \vec{E}_2(\vec{r}_2)
\]

will have to be evaluated with \( \vec{r}_2 \to \vec{r}_p \) from region 2 along the dotted normal vector in Figure 14. This then gives

\[
\lim_{\vec{r}_2 \to \vec{r}_p} \varepsilon_{r_2} \hat{n}_1 \cdot \vec{E}_2(\vec{r}_2) = 2 \varepsilon_{r_2} \left[ \hat{n}_1 \cdot \vec{E}_{inc}(\vec{r}_p) + \hat{n}_1 \cdot \frac{1}{4\pi} \int_{S_{14}} \left[ j \omega \mu_1 (\hat{n}'_1 \times \vec{H}_1) \frac{R_1}{R_1} - (\hat{n}'_1 \cdot \vec{E}_1) \frac{R_1}{R_1} \right] d\vec{s}' + \hat{n}_1 \cdot \frac{1}{4\pi} \int_{S_{24}} \left[ j \omega \mu_2 (\hat{n}'_2 \times \vec{H}_2) \frac{R_2}{R_2} - (\hat{n}'_2 \cdot \vec{E}_2) \frac{R_2}{R_2} \right] d\vec{s}' + \frac{-1}{4\pi} \left( \frac{\varepsilon_{r_1} - \varepsilon_{r_2}}{\varepsilon_{r_2}} \right) \lim_{\vec{r}_2 \to \vec{r}_p} \hat{n}_1 \cdot \int_{S_{12}} (\hat{n}'_1 \cdot \vec{E}_1) \frac{R}{R^3} d\vec{s}' + \frac{-1}{4\pi} j \omega (\mu_1 - \mu_2) \lim_{\vec{r}_2 \to \vec{r}_p} \hat{n}_1 \cdot \int_{S_{12}} (\hat{n}'_1 \times \vec{H}_1) \frac{R}{R^3} d\vec{s}' \right].
\]

(4.33)

Similarly, we have [10]-[11]

\[
\lim_{\vec{r}_2 \to \vec{r}_p} \hat{n}_1 \cdot \int_{S_{12}} (\hat{n}'_1 \cdot \vec{E}_1) \frac{R}{R^3} d\vec{s}' = -2\pi (\hat{n}_1 \cdot \vec{E}_1(\vec{r}_p)) + \hat{n}_1 \cdot \int_{S_{12}} (\hat{n}'_1 \times \vec{H}_1) \frac{R_p}{R_p} d\vec{s}'.
\]

(4.34)

Notice that the term \(-2\pi (\hat{n}_1 \cdot \vec{E}_1(\vec{r}_p))\) has a sign that is opposite to the second term in (4.30). This is because \( \vec{p} \) is approached from a different direction. We also have [18]

\[
\lim_{\vec{r}_2 \to \vec{r}_p} \hat{n}_1 \cdot \int_{S_{12}} (\hat{n}'_1 \times \vec{H}_1) \frac{1}{R} d\vec{s}' = \hat{n}_1 \cdot \int_{S_{12}} (\hat{n}'_1 \times \vec{H}_1) \frac{1}{R_p} d\vec{s}'.
\]

(4.35)
This then gives

\[
\lim_{\epsilon_2 \to \epsilon_1} \epsilon_{r_1} \hat{n}_1 \cdot \vec{E}_2(\vec{r}_2) = 2 \epsilon_{r_2} \left[ \hat{n}_1 \cdot \vec{E}_1(\vec{r}_p) + \right.
\]

\[
\left. \hat{n}_1 \cdot \frac{-1}{4\pi} \int_{S_{14}} \left[ j\omega \mu_1 (\hat{n}_1' \times \vec{H}_1) \frac{1}{R_1} - (\hat{n}_1' \cdot \vec{E}_1) \frac{\vec{R}_1}{R_1^3} \right] ds' + \right.
\]

\[
\left. \hat{n}_1 \cdot \frac{-1}{4\pi} \int_{S_{24}} \left[ j\omega \mu_2 (\hat{n}_2' \times \vec{H}_2) \frac{1}{R_2} - (\hat{n}_2' \cdot \vec{E}_2) \frac{\vec{R}_2}{R_2^3} \right] ds' + \right.
\]

\[
\left. \frac{-1}{4\pi} \left( \frac{\epsilon_{r_1} - \epsilon_{r_2}}{\epsilon_{r_2}} \right) \left( -2\pi (\hat{n}_1 \cdot \vec{E}_1(\vec{r}_p)) + \hat{n}_1 \cdot \int_{S_{12}} (\hat{n}_1' \cdot \vec{E}_1) \frac{\vec{R}_p}{R_p^3} ds' \right) + \right.
\]

\[
\left. \hat{n}_1 \cdot \frac{-1}{4\pi} j\omega (\mu_1 - \mu_2) \int_{S_{12}} (\hat{n}_1' \times \vec{H}_1) \frac{1}{R_p} ds' \right]
\]

\[
= \epsilon_{r_2} \hat{n}_1 \cdot \vec{E}_2(\vec{r}_p)
\]  

(4.36)

where, again, \( R_1 = \vec{r}_p - \vec{r}_{14}, R_2 = \vec{r}_p - \vec{r}_{24}, \) and \( R_p = \vec{r}_p - \vec{r}_{12}. \) Now to enforce the boundary condition in (4.28), (4.36) can be subtracted from (4.32) on \( S_{12} \) giving

\[
\hat{n}_1 \cdot (\epsilon_{r_1} \vec{E}_1(\vec{r}_p) - \epsilon_{r_2} \vec{E}_2(\vec{r}_p))
\]

\[
= 2 \left[ \hat{n}_1 \cdot (\epsilon_{r_1} - \epsilon_{r_2}) \vec{E}_1(\vec{r}_p) + \right.
\]

\[
\left. \hat{n}_1 \cdot \frac{-1}{4\pi} \epsilon_{r_1} \int_{S_{14}} \left[ j\omega \mu_1 (\hat{n}_1' \times \vec{H}_1) \frac{1}{R_1} - (\hat{n}_1' \cdot \vec{E}_1) \frac{\vec{R}_1}{R_1^3} \right] ds' + \right.
\]

\[
\left. \hat{n}_1 \cdot \frac{-1}{4\pi} \epsilon_{r_1} \int_{S_{24}} \left[ j\omega \mu_2 (\hat{n}_2' \times \vec{H}_2) \frac{1}{R_2} - (\hat{n}_2' \cdot \vec{E}_2) \frac{\vec{R}_2}{R_2^3} \right] ds' + \right.
\]

\[
\left. \frac{-2\pi}{4\pi} \left( \frac{\epsilon_{r_1} - \epsilon_{r_2}}{\epsilon_{r_2}} \right) (\epsilon_{r_1} + \epsilon_{r_2}) (\hat{n}_1 \cdot \vec{E}_1(\vec{r}_p)) + \right.
\]

\[
\left. \frac{-1}{4\pi} \epsilon_{r_1} \left( \frac{\epsilon_{r_1} - \epsilon_{r_2}}{\epsilon_{r_2}} \right) (\epsilon_{r_1} - \epsilon_{r_2}) \hat{n}_1 \cdot \int_{S_{12}} (\hat{n}_1' \cdot \vec{E}_1) \frac{\vec{R}_p}{R_p^3} ds' + \right.
\]

\[
\left. \frac{-1}{4\pi} j\omega (\mu_1 - \mu_2) (\epsilon_{r_1} - \epsilon_{r_2}) \hat{n}_1 \cdot \int_{S_{12}} (\hat{n}_1' \times \vec{H}_1) \frac{1}{R_p} ds' \right]
\]

\[
= 0.
\]  

(4.37)
Factoring out a \((\varepsilon_{r_1} - \varepsilon_{r_2})\) and simplifying, we get

\[
\hat{n}_1 \cdot (\varepsilon_{r_1} \hat{E}_1(\vec{r}_p) - \varepsilon_{r_2} \hat{E}_2(\vec{r}_p))
\]

\[
= (\varepsilon_{r_1} - \varepsilon_{r_2}) \left[ \hat{n}_1 \cdot \hat{E}_{inc}(\vec{r}_p) + \hat{n}_1 \cdot \frac{-1}{4\pi} \int_{S_{14}} \left[ j\omega \mu_1 (\hat{n}'_1 \times \hat{H}_1) \frac{1}{R_1} - (\hat{n}'_1 \cdot \hat{E}_1) \frac{\bar{R}_1}{R_1^3} \right] ds' + \right.
\]
\[
\hat{n}_1 \cdot \frac{-1}{4\pi} \int_{S_{24}} \left[ j\omega \mu_2 (\hat{n}'_2 \times \hat{H}_2) \frac{1}{R_2} - (\hat{n}'_2 \cdot \hat{E}_2) \frac{\bar{R}_2}{R_2^3} \right] ds' + \right.
\]
\[
\left. \frac{-1}{2} \left( \frac{\varepsilon_{r_1} + \varepsilon_{r_2}}{\varepsilon_{r_2}} \right) \left( \hat{n}_1 \cdot \hat{E}_1(\vec{r}_p) \right) + \right. 
\]
\[
\left. \frac{-1}{4\pi} \int_{S_{12}} \left( \hat{n}'_1 \cdot \hat{E}_1 \right) \frac{\bar{R}_p}{R_p^3} ds' + \right. 
\]
\[
\left. \frac{-1}{4\pi} j\omega (\mu_1 - \mu_2) \hat{n}_1 \cdot \int_{S_{12}} \left( \hat{n}'_1 \times \hat{H}_1 \right) \frac{1}{R_p} ds' \right]. 
\]

\[
= 0. 
\]

Solving for \(\hat{E}_{inc}(\vec{r}_p)\) gives

\[
\hat{n}_1 \cdot \hat{E}_{inc}(\vec{r}_p) = \hat{n}_1 \cdot \frac{1}{4\pi} \int_{S_{14}} \left[ j\omega \mu_1 (\hat{n}'_1 \times \hat{H}_1) \frac{1}{R} - (\hat{n}'_1 \cdot \hat{E}_1) \frac{\bar{R}}{R^3} \right] ds' + 
\]
\[
\hat{n}_1 \cdot \frac{1}{4\pi} \int_{S_{24}} \left[ j\omega \mu_2 (\hat{n}'_2 \times \hat{H}_2) \frac{1}{R} - (\hat{n}'_2 \cdot \hat{E}_2) \frac{\bar{R}}{R^3} \right] ds' + 
\]
\[
\left( \frac{\varepsilon_{r_1} + \varepsilon_{r_2}}{2\varepsilon_{r_2}} \right) \left( \hat{n}_1 \cdot \hat{E}_1(\vec{r}_p) \right) + 
\]
\[
\hat{n}_1 \cdot \frac{\varepsilon_{r_1} - \varepsilon_{r_2}}{4\pi \varepsilon_{r_2}} \int_{S_{12}} \left( \hat{n}'_1 \cdot \hat{E}_1 \right) \frac{\bar{R}}{R^3} ds' + 
\]
\[
\hat{n}_1 \cdot j\omega (\mu_1 - \mu_2) \frac{1}{4\pi} \int_{S_{12}} (\hat{n}'_1 \times \hat{H}_1) \frac{1}{R} ds'. 
\]

Notice the subscripts for \(R\) and \(\bar{R}\) have been dropped. (4.38) is very similar to (2.35) except that (4.38) has the inductive terms present in the integrals over \(S_{14}, S_{24},\) and \(S_{12}\). Equation (4.38) can be written in terms of total surface charge density \(\rho_{tot}\) and surface current density \(\bar{J}_s\) only. The expressions for \(\bar{J}_s\) and \(\rho_{tot}\) used here have been
evaluated in Section 2.3.2. A summary of these derivations is provided here. Since $S_{12}$ is a dielectric/dielectric interface, the normal components of the electric field are related through
\[
\hat{n}_1 \cdot [\varepsilon_1 \vec{E}_1(\vec{r}) - \varepsilon_2 \vec{E}_2(\vec{r})] = \rho_{\text{free}}(\vec{r}) \tag{4.39}
\]
and
\[
-\hat{n}_1 \cdot [\vec{P}_1(\vec{r}) - \vec{P}_2(\vec{r})] = \rho_{\text{pol}}(\vec{r}). \tag{4.40}
\]
Since $\rho_{\text{free}} = 0$, we can solve for the electric field in region two as
\[
\hat{n}_1 \cdot \vec{E}_2(\vec{r}) = \frac{\varepsilon_{r_1}}{\varepsilon_{r_2}} \hat{n}_1 \cdot \vec{E}_1(\vec{r}). \tag{4.41}
\]
Using (4.40) and (4.41), the following relation for $\rho_{\text{pol}}$ can be written as
\[
\left( \frac{\varepsilon_{r_2} - \varepsilon_{r_1}}{\varepsilon_{r_2}} \right) \hat{n}_1 \cdot \vec{E}_1(\vec{r}) = \frac{\rho_{\text{pol}}(\vec{r})}{\varepsilon_0} = \frac{\rho_{\text{tot}}(\vec{r})}{\varepsilon_0}. \tag{4.42}
\]
Next, on the conductor/dielectric interfaces of $S_{14}$ and $S_{24}$, the normal components of the electric field are related to the charge by
\[
\hat{n}_1 \cdot \vec{E}_1(\vec{r}) = \frac{\rho_{\text{free}}}{\varepsilon_0} + \frac{\rho_{\text{pol}}(\vec{r})}{\varepsilon_0} = \frac{\rho_{\text{tot}}(\vec{r})}{\varepsilon_0} \tag{4.43}
\]
and
\[
\hat{n}_2 \cdot \vec{E}_2(\vec{r}) = \frac{\rho_{\text{free}}}{\varepsilon_0} + \frac{\rho_{\text{pol}}(\vec{r})}{\varepsilon_0} = \frac{\rho_{\text{tot}}(\vec{r})}{\varepsilon_0}. \tag{4.44}
\]
Finally, on $S_{14}$, $S_{24}$, and $S_{12}$, the tangential components of the magnetic field are related to the surface current density by [1]
\[
\hat{n}_1 \times \vec{H}_1 = \vec{J}_s \tag{4.45}
\]
57
and

$$\hat{n}_2 \times \vec{H}_2 = \vec{J}_s.$$  \hfill (4.46)

Now, rearranging (4.38) gives

$$\left(\frac{\varepsilon_{r_1} + \varepsilon_{r_2}}{2\varepsilon_{r_2}}\right)\left(\frac{\varepsilon_{r_2} - \varepsilon_{r_1}}{\varepsilon_{r_2} - \varepsilon_{r_1}}\right)(\hat{n}_1 \cdot \vec{E}_1(\bar{r}_p)) = \hat{n}_1 \cdot \vec{E}_{inc}(\bar{r}_p) -$$

$$\hat{n}_1 \cdot \frac{1}{4\pi} \int_{S_{14}} j\omega \mu_1 (\hat{n}'_1 \times \vec{H}_1) \frac{1}{R} ds' +$$

$$\hat{n}_1 \cdot \frac{1}{4\pi} \int_{S_{14}} (\hat{n}'_1 \cdot \vec{E}_1) \frac{\bar{R}}{R^3} ds' -$$

$$\hat{n}_1 \cdot \frac{1}{4\pi} \int_{S_{24}} j\omega \mu_2 (\hat{n}'_2 \times \vec{H}_2) \frac{1}{R} ds' +$$

$$\hat{n}_1 \cdot \frac{1}{4\pi} \int_{S_{24}} (\hat{n}'_2 \cdot \vec{E}_2) \frac{\bar{R}}{R^3} ds' -$$

$$\hat{n}_1 \cdot \frac{1}{4\pi \varepsilon_{r_2}} \int_{S_{12}} \rho_s(\bar{r}') \frac{\bar{R}}{R^3} ds' -$$

$$\hat{n}_1 \cdot j\omega (\mu_1 - \mu_2) \frac{1}{\varepsilon_0} \int_{S_{12}} (\hat{n}'_1 \times \vec{H}_1) \frac{1}{R} ds'.$$

Multiplying both sides by \((\varepsilon_{r_2} - \varepsilon_{r_1})\) and substituting (4.42)-(4.46) gives

$$\left(\frac{\varepsilon_{r_1} + \varepsilon_{r_2}}{2\varepsilon_0}\right) \rho_{tot}(\bar{r}_p) = \hat{n}_1(\varepsilon_{r_2} - \varepsilon_{r_1}) \cdot \vec{E}_{inc}(\bar{r}_p) +$$

$$\hat{n}_1 \cdot (\varepsilon_{r_2} - \varepsilon_{r_1}) \frac{j\omega \mu_1}{4\pi} \int_{S_{14}} \vec{J}_s(\bar{r}') \frac{1}{R} ds' +$$

$$\hat{n}_1 \cdot (\varepsilon_{r_2} - \varepsilon_{r_1}) \frac{j\omega \mu_2}{4\pi} \int_{S_{24}} \vec{J}_s(\bar{r}') \frac{1}{R} ds' -$$

$$\hat{n}_1 \cdot (\varepsilon_{r_1} - \varepsilon_{r_2}) \frac{\rho_s(\bar{r}')}{4\pi} \frac{\bar{R}}{R^3} ds' -$$

$$\hat{n}_1 \cdot (\varepsilon_{r_1} - \varepsilon_{r_2}) \frac{\rho_s(\bar{r}')}{4\pi} \frac{\bar{R}}{R^3} ds' -$$

$$\hat{n}_1 \cdot (\varepsilon_{r_1} - \varepsilon_{r_2}) \frac{\rho_s(\bar{r}')}{4\pi} \frac{\bar{R}}{R^3} ds' +$$

$$\hat{n}_1 \cdot (\varepsilon_{r_1} - \varepsilon_{r_2}) \frac{j\omega (\mu_1 - \mu_2)}{4\pi} \int_{S_{12}} \vec{J}_s(\bar{r}') \frac{1}{R} ds'. \hfill (4.47)
Using (4.45) and the current continuity equation, $\vec{E}_{\text{inc}}(\vec{r}_p)$ can be written as

$$
\vec{E}_{\text{inc}}(\vec{r}_p) = \frac{-1}{4\pi} \int_{S_{13}} j\omega \mu_1 \vec{J}_s(\vec{r}')\phi_1 d\vec{s}' + \nabla \frac{1}{4\pi j\omega \varepsilon_0} \int_{S_{13}} \nabla' \cdot \vec{J}_s(\vec{r}')\phi_1 d\vec{s}'.
$$

(4.48)

Equation (4.47) can be written in a slightly more compact form of:

$$
\left( \frac{\varepsilon_{r1} + \varepsilon_{r2}}{2\varepsilon_0} \right) \rho_{\text{tot}}(\vec{r}_p) = -(\varepsilon_{r1} - \varepsilon_{r2}) \hat{n}_1 \cdot \vec{E}_{\text{inc}}(\vec{r}_p) + \frac{(\varepsilon_{r1} - \varepsilon_{r2})}{4\pi} \hat{n}_1 \cdot j\omega \mu_i \int_{\sum \partial_i^c} \vec{J}_n(\vec{r}') \frac{1}{R} d\vec{s}' - \frac{(\varepsilon_{r1} - \varepsilon_{r2})}{4\pi} \hat{n}_1 \cdot \int_{\sum \partial_i^d} \rho_s(\vec{r}') \frac{R}{R^3} d\vec{s}' + \frac{(\varepsilon_{r1} - \varepsilon_{r2})}{4\pi} j\omega (\mu_1 - \mu_2) \hat{n}_1 \cdot \int_{S_{12}} \vec{J}_{s12}(\vec{r}') \frac{1}{R} d\vec{s}'.
$$

(4.49)

where $\sum \partial_i^c$ is the collection of all conductor/dielectric interfaces in the quasi-static regions. Again, it should be noted that $\vec{E}_{\text{inc}}$ can also be used to evaluate the contribution from other quasi-static regions. Equation (4.49) is very similar to (2.73), except for the present inductive terms. Equation (4.49) is the first equation with two unknowns: charge and current. This now completes our derivation for the dielectric/dielectric boundary condition on $S_{12}$.

4.3.3. Conductor/Dielectric Interfaces

The next boundary condition enforced will be on a conductor/dielectric interface. This boundary exists on $S_{14}$ and $S_{24}$. Choosing $S_{14}$, the boundary condition for the electric field is

$$
\hat{n}_1 \times \vec{E}_1(\vec{r}_1) = 0.
$$

(4.50)
where \( \vec{r}_1 \) is the vector on \( S_{14} \). Using equation (4.25) in (4.50) gives

\[
\hat{n}_1 \times \vec{E}_1(\vec{r}) = 0 = \hat{n}_1 \times 2 \left[ \vec{E}_{\text{inc}} + \right.
\begin{align*}
&\frac{-1}{4\pi} \int_{S_{14}} j \omega \mu_1 (\hat{n}'_1 \times \vec{H}_1) \frac{1}{R} ds' + \\
&\frac{-1}{4\pi} \int_{S_{24}} j \omega \mu_2 (\hat{n}'_2 \times \vec{H}_2) \frac{1}{R} ds' + \\
&\frac{-1}{4\pi} \int_{S_{12}} j \omega (\mu_1 - \mu_2) (\hat{n}'_1 \times \vec{H}_1) \frac{1}{R} ds' + \\
&\frac{1}{4\pi} \frac{\varepsilon_{r_2} - \varepsilon_{r_1}}{\varepsilon_{r_2}} \int_{S_{12}} (\hat{n}'_1 \cdot \vec{E}_1) \frac{R}{R^3} ds'.
\end{align*}
\]

Then, by (4.5) we have

\[
\hat{n}_1 \times \vec{E}_1(\vec{r}) = 0 = \hat{n}_1 \times 2 \left[ \vec{E}_{\text{inc}} + \right.
\begin{align*}
&\frac{-1}{4\pi} \int_{S_{14}} j \omega \mu_1 (\hat{n}'_1 \times \vec{H}_1) \frac{1}{R} ds' - \nabla \frac{1}{4\pi} \int_{S_{14}} (\hat{n}'_1 \cdot \vec{E}_1) \frac{1}{R} ds' + \\
&\frac{-1}{4\pi} \int_{S_{24}} j \omega \mu_2 (\hat{n}'_2 \times \vec{H}_2) \frac{1}{R} ds' - \nabla \frac{1}{4\pi} \int_{S_{24}} (\hat{n}'_2 \cdot \vec{E}_2) \frac{1}{R} ds' + \\
&\frac{-1}{4\pi} \int_{S_{12}} j \omega (\mu_1 - \mu_2) (\hat{n}'_1 \times \vec{H}_1) \frac{1}{R} ds' + \\
&\nabla \frac{-1}{4\pi} \frac{\varepsilon_{r_2} - \varepsilon_{r_1}}{\varepsilon_{r_2}} \int_{S_{12}} (\hat{n}'_1 \cdot \vec{E}_1) \frac{1}{R} ds'.
\end{align*}
\]

\Rightarrow

\[
\hat{n}_1 \times \vec{E}_1(\vec{r}) = 0 = \hat{n}_1 \times (-j \omega \vec{A}_{\text{tot}} - \nabla \psi_{\text{tot}} + \vec{E}_{\text{inc}})
\]

where

\[
\vec{E}_{\text{inc}} = \frac{-1}{4\pi} \int_{S_{13}} j \omega \mu_1 (\hat{n}'_1 \times \vec{H}_1) \phi_1 ds' - \nabla \frac{1}{4\pi} \int_{S_{13}} (\hat{n}'_1 \cdot \vec{E}_1) \phi_1 ds',
\]\n
60
\[\bar{A}_{\text{tot}} = \frac{1}{4\pi} \int_{S_{14}} \mu_1 (\hat{n}_1' \times \bar{H}_1) \frac{1}{R} ds' + \frac{1}{4\pi} \int_{S_{24}} \mu_2 (\hat{n}_2' \times \bar{H}_2) \frac{1}{R} ds' + \frac{1}{4\pi} \int_{S_{12}} (\mu_1 - \mu_2) (\hat{n}_1' \times \bar{H}_1) \frac{1}{R} ds', \quad (4.55)\]

and

\[\psi_{\text{tot}} = \frac{1}{4\pi} \int_{S_{14}} (\hat{n}_1' \cdot \bar{E}_1) \frac{1}{R} ds' + \frac{1}{4\pi} \int_{S_{24}} (\hat{n}_2' \cdot \bar{E}_2) \frac{1}{R} ds' + \frac{1}{4\pi} \int_{S_{12}} (\varepsilon_{r2} - \varepsilon_{r1}) \frac{1}{\bar{\varepsilon}_r} (\hat{n}_1' \cdot \bar{E}_1) \frac{1}{R} ds'. \quad (4.56)\]

Equation (4.53) can be written in terms of charges and currents only. Using (4.45) and (4.46), (4.55) can be written as

\[\bar{A}_{\text{tot}} = \frac{1}{4\pi} \int_{S_{14}} \mu_1 \bar{J}_s(\bar{r}') \frac{1}{R} ds' + \frac{1}{4\pi} \int_{S_{24}} \mu_2 \bar{J}_s(\bar{r}') \frac{1}{R} ds' + \frac{1}{4\pi} \int_{S_{12}} (\mu_1 - \mu_2) \bar{J}_s(\bar{r}') \frac{1}{R} ds'. \quad (4.57)\]

Using (4.42) and (4.43), (4.56) can be written as

\[\psi_{\text{tot}} = \frac{1}{4\pi} \int_{S_{14}} \rho_s(\bar{r}') \frac{1}{\varepsilon_0} \frac{1}{R} ds' + \frac{1}{4\pi} \int_{S_{24}} \rho_s(\bar{r}') \frac{1}{\varepsilon_0} \frac{1}{R} ds' + \frac{1}{4\pi} \int_{S_{12}} \rho_s(\bar{r}') \frac{1}{\varepsilon_0} \frac{1}{R} ds'. \quad (4.58)\]

Using (4.43), (4.45), and \(-j\omega \rho_s = \nabla' \cdot \bar{J}_s\) the expression for the incident field can be
written as

\[ \vec{E}_{\text{inc}} = -\frac{1}{4\pi} \int_{S_{13}} j\omega\mu_1 \vec{J}_s(\vec{r}') \phi_1 d\vec{s}' + \nabla \frac{1}{4\pi j\omega\varepsilon_0} \int_{S_{13}} \nabla' \cdot \vec{J}_s(\vec{r}') \phi_1 d\vec{s}'. \]  

(4.59)

Equations (4.57) and (4.58) can be written in the simpler form of

\[ \vec{A}_{\text{tot}} = \frac{1}{4\pi} \int_{\sum_{\sigma_i}} \mu_i \vec{J}_s(\vec{r}') \frac{1}{R} d\vec{s}' + \frac{1}{4\pi} \int_{S_{12}} (\mu_1 - \mu_2) \vec{J}_s(\vec{r}') \frac{1}{R} d\vec{s}' \]  

(4.60)

and

\[ \psi_{\text{tot}} = \frac{1}{4\pi \varepsilon_0} \int_{\sum_{\sigma_i^Q}} \frac{\rho_s(\vec{r}')}{R} d\vec{s}'. \]  

(4.61)

Equation (4.60) is the term that represents the inductive effects in the quasi-static region. This term is the same term as the neglected term (6) by Olsen, Hower, and Mannikko. Now we have a second equation in terms of two unknowns: charge and current. This ends the derivation for the EFIE on the conductor/dielectric interface of \( S_{14} \). A similar derivation can be taken by enforcing the boundary conditions on \( S_{24} \). The unknowns are in both the quasi-static regions and the full-wave regions. This means two more equations are needed. All types of boundary conditions in the quasi-static region are now exhausted. Now the full-wave field points will be considered.

4.4. Full-Wave Equations

The derivations for the full-wave EFIE are exactly like the steps found in Section 2.4. A summary of these steps will be presented here.

4.4.1. General Expression for the Electric Field

Using the equivalence principle, the expression for the electric field in region 1 can be written as

\[ \vec{E}_1(\vec{r}) = \frac{-1}{4\pi} \int_{S_{13}} j\omega\mu_1(\hat{n}'_1 \times \vec{H}_1) \phi_1 d\vec{s}' + \frac{1}{4\pi} \int_{S_{13}} (\hat{n}'_1 \cdot \vec{E}_1) \nabla' \phi_1 d\vec{s}' + \vec{E}^Q(\rho, z). \]  

(4.62)
\( \vec{E}^Q(\rho, z) \) is the expression for the electric field from sources on \( S_{12}, S_{14}, \) and \( S_{24} \) in the quasi-static region and can be written as

\[
\vec{E}^Q(\rho, z) = -j\omega \vec{A}_i - \nabla \psi_i
\]  

(4.63)

with

\[
\vec{A}_i = \frac{1}{4\pi} \int_{\Sigma^Q} \mu_1 (\hat{n}'_i \times \vec{H}_i) \phi_i ds'
\]

(4.64)

and

\[
\psi_i = \frac{1}{4\pi} \int_{\Sigma^Q} (\hat{n}'_i \cdot \vec{E}_i) \phi_i ds'
\]

(4.65)

where \( i=1,2. \)

4.4.2. Conductor/Dielectric Interfaces

Using the boundary condition \( \hat{n}_1 \times \vec{E}_1(\vec{r}) = 0 \) on \( S_{13} \) in Figure 13, the expression for the electric field can be written as

\[
\hat{n}_1 \times \vec{E}_1(\vec{r}) = 0
\]

\[
= \hat{n}_1 \times \left[ -\frac{1}{4\pi} \int_{S_{13}} j\omega \mu_1 (\hat{n}'_1 \times \vec{H}_1) \phi_1 ds' + \nabla' \left( \frac{1}{4\pi} \int_{S_{13}} (\hat{n}'_1 \cdot \vec{E}_1) \phi_1 ds' \right) \\
+ \vec{E}^Q(\rho, z) \right].
\]

(4.66)

Now (4.66) can be written in terms of charge and current only. Using (4.43)-(4.46) and \( -j\omega \rho_s = \nabla' \cdot \vec{J}_s, \) (4.66) can be written as

\[
\hat{n}_1 \times \vec{E}_1(\vec{r}) = 0
\]

(4.67)

where

\[
\vec{E}_1(\vec{r}) = -\frac{1}{4\pi} \int_{S_{13}} j\omega \mu_1 \vec{J}_s(\vec{r}') \phi_1 ds' + \nabla' \left( \frac{1}{4\pi j\omega \varepsilon_0} \int_{S_{13}} \nabla' \cdot \vec{J}_s(\vec{r}') \phi_1 ds' \right) + \vec{E}^Q(\rho, s)
\]

(4.68)
and

\[ E^Q(\rho, s) = \frac{-1}{4\pi} \int_{\sum \partial_R} j\omega \mu_1 \tilde{J}_s(\vec{r}') \phi_i ds' - \nabla \frac{1}{4\pi \varepsilon_0} \int_{\sum \partial_R} \rho_s(\vec{r}') \phi_i ds'. \] (4.69)

Equation (4.67) gives us a third equation in terms of two unknowns: charge and current. This ends the derivation for the EFIE on the conductor/dielectric interface in the full-wave region. All types of boundary conditions in the full-wave region are now evaluated and only one more equation is needed to provide enough equations to solve for the unknown charges and currents.

4.5. Current Continuity Derivations

The process of deriving the current continuity equations is exactly like Section 2.5. A summary of this process will be presented here. The conservation of charge is enforced through

\[ \varepsilon_{r_1} \int_{\sum \partial_F} \rho_s(\vec{r}') ds' + \varepsilon_{r_i} \int_{\sum \partial_i} \rho_s(\vec{r}') ds' = K \] (4.70)

and the current continuity, in general, is

\[ \nabla' \cdot \bar{J}_s(\vec{r}') = -j\omega \rho_s(\vec{r}') \] (4.71)

where, again, \( K \) is the total charge in a region and is still assumed to be zero for our problems. By multiplying both sides of (4.70) by \( j\omega \) and substituting (4.71) for the first term we get

\[ \bar{J}(\vec{r}_a) - \bar{J}(\vec{r}_b) = -j\omega\varepsilon_{r_i} \int_{\sum \partial_i} \rho_s(\vec{r}') ds' \] (4.72)

where \( \bar{J} \) is the current along the full-wave surface between the limits of integration \( \vec{r}_a \) and \( \vec{r}_b \). (4.72) gives the final relation needed between current and charge in both the quasi-static region and full-wave regions.
4.6. Summary of the New EFIE

The following is a summary of previously derived equations. For match points on the dielectric/dielectric interfaces in the quasi-static regions

\[
\left(\frac{\varepsilon_{r_1} + \varepsilon_{r_2}}{2\varepsilon_0}\right)\rho_{tot}(\vec{r}_p) = -(\varepsilon_{r_1} - \varepsilon_{r_2})\hat{n}_1 \cdot \vec{E}_{inc}(\vec{r}_p) + \\
\frac{(\varepsilon_{r_1} - \varepsilon_{r_2})}{4\pi} \hat{n}_1 \cdot j\omega\mu_i \int_{\partial\sigma_i} \vec{J}_s(\vec{r}') \frac{1}{R} ds' - \\
\frac{(\varepsilon_{r_1} - \varepsilon_{r_2})}{4\pi} \hat{n}_1 \cdot \int_{\partial\sigma_i} \frac{\rho_s(\vec{r}')}{\varepsilon_0} \frac{R}{R^3} ds' + \\
\frac{(\varepsilon_{r_1} - \varepsilon_{r_2})}{4\pi} j\omega(\mu_1 - \mu_2)\hat{n}_1 \cdot \int_{S_{12}} \vec{J}_{s12}(\vec{r}') \frac{1}{R} ds' \quad (4.73)
\]

where

\[
\vec{E}_{inc}(\vec{r}_p) = -\frac{1}{4\pi} \int_{S_{13}} j\omega\mu_1 \vec{J}_s(\vec{r}')\phi_1 ds' + \nabla \frac{1}{4\pi j\omega \varepsilon_0} \int_{S_{13}} \nabla' \cdot \vec{J}_s(\vec{r}') \phi_1 ds'. \quad (4.74)
\]

For match points on the conductor/dielectric interfaces in the quasi-static region we have

\[
\hat{n}_1 \times \vec{E}_1(\vec{r}) = 0 = \hat{n}_1 \times (-j\omega \vec{A}_{tot} - \nabla \psi_{tot} + \vec{E}_{inc}) \quad (4.75)
\]

where

\[
\vec{A}_{tot} = \frac{1}{4\pi} \int_{\sigma_i} \mu_i \vec{J}_s(\vec{r}') \frac{1}{R} ds' + \frac{1}{4\pi} \int_{S_{12}} (\mu_1 - \mu_2)\vec{J}_s(\vec{r}') \frac{1}{R} ds', \\
\psi_{tot} = \frac{1}{4\pi \varepsilon_0} \int_{\sigma_i} \frac{\rho_s(\vec{r}')}{R} ds', \quad (4.76)
\]

and

\[
\vec{E}_{inc} = -\frac{1}{4\pi} \int_{S_{13}} j\omega\mu_1 \vec{J}_s(\vec{r}')\phi_1 ds' + \nabla \frac{1}{4\pi j\omega \varepsilon_0} \int_{S_{13}} \nabla' \cdot \vec{J}_s(\vec{r}') \phi_1 ds'. \quad (4.78)
\]
For match points on the conductor/dielectric interface in the full-wave region we have

\[ \hat{n}_1 \times \vec{E}_1(\vec{r}) = 0 \]  \hspace{1cm} (4.79)

where

\begin{align*}
\vec{E}_1(\vec{r}) &= -\frac{1}{4\pi} \int_{S_{13}} j\omega\mu_1 \vec{J}_s(\vec{r}') \phi_1 ds' + \nabla \cdot \left( \frac{1}{4\pi j\omega \varepsilon_0} \int_{S_{13}} \nabla' \cdot \vec{J}_s(\vec{r}') \phi_1 ds' \right) + \vec{E}^Q(\rho, s) \hspace{1cm} (4.80)
\end{align*}

and

\begin{align*}
\vec{E}^Q(\rho, s) &= -\frac{1}{4\pi} \int_{\sum \phi_i} j\omega\mu_1 \vec{J}_s(\vec{r}') \phi_1 ds' - \nabla \cdot \left( \frac{1}{4\pi \varepsilon_0} \int_{\sum \phi_i} \rho_s(\vec{r}') \phi_1 ds' \right). \hspace{1cm} (4.81)
\end{align*}

Finally, the current continuity equation is

\[ \vec{J}(\vec{r}_a) - \vec{J}(\vec{r}_b) = -j\omega \varepsilon_r \int_{\sum \phi_i} \rho_s(\vec{r}') ds'. \hspace{1cm} (4.82) \]

Now we have enough integral equations to solve for the desired unknowns.
CHAPTER 5. IMPLEMENTATION OF THE NEW EFIE

5.1. Discrete Version of the Quasi-Static Equations

The pulse definitions outlined in Section 3.2 are implemented in this section. For all the calculations in this chapter, it is assumed that thin-wire assumptions exist in all regions. Now we will take a look at the EFIE with match points in the quasi-static region. To solve these integral equations, the unknown charge will have to be expanded into known expansion functions and unknown charge pulses. Using (3.9), we get

$$\rho_s = \sum_n u_n \rho_n$$  \hspace{1cm} (5.1)

for the expansion of unknown charge. The unknown current will also be expanded into known expansion functions and unknown current pulses. This then gives

$$\bar{I} = \sum_n v_n I_n \hat{v}_n$$  \hspace{1cm} (5.2)

where $\hat{v}_n$ is the unit vector at the current pulse and $v_n$ is given in (3.10). The first integral equation considered is (4.73). Using (5.1) and (5.2), we can write

$$\left(\frac{\varepsilon_{r1} + \varepsilon_{r2}}{2\varepsilon_0}\right) \rho_m = - (\varepsilon_{r1} - \varepsilon_{r2}) \bar{E}_{inc}(\vec{r}) +$$

$$\frac{(\varepsilon_{r1} - \varepsilon_{r2})}{4\pi j \omega} \hat{n}_1 \cdot \sum_n \hat{v}_n \int_{s_{n+}} \mu_n v_n I_n \frac{1}{R} ds' -$$

$$\frac{(\varepsilon_{r1} - \varepsilon_{r2})}{4\pi \varepsilon_0} \hat{n}_1 \cdot \sum_n \int_{s_n} u_n \rho_n \frac{\bar{R}}{R^3} ds' +$$

$$\frac{(\varepsilon_{r1} - \varepsilon_{r2})}{4\pi j \omega (\mu_1 - \mu_2)} \hat{n}_1 \cdot \hat{v}' \int_{s_{n+}} v_n I_n \frac{1}{R_p} ds'$$  \hspace{1cm} (5.3)

where

$$\bar{E}_{inc} = \frac{-1}{4\pi} \left[ \hat{n}_1 \cdot \hat{z}' \sum_n \int_{s_{n+}} j \omega \mu_1 v_n I_n \phi_1 dz' - \hat{z}' \frac{1}{j \omega \varepsilon_0} \frac{\partial}{\partial n} \sum_n \frac{I_{n+1} - I_n}{\Delta z'} \int_{s_n} v_n \phi_1 dz' \right].$$
\( \frac{\partial}{\partial n} \) is the partial derivative in the normal direction and \( \hat{z}' \) is the unit vector at the source point in the full-wave region. Notice the integrals in (5.3) are evaluated over only the unit pulses that contain the source points of interest. Similarly, for match points on the conductor/dielectric interface, (4.75) can be written as

\[
-j\omega \vec{A}_{\text{tot}} - \nabla \psi_{\text{tot}} + \vec{E}_{\text{inc}} = 0
\]

where \( \vec{E}_{\text{inc}} \) is given in (5.3) with \( \frac{\partial}{\partial n} \) replaced by \( \frac{\partial}{\partial s} \).

\[
\vec{A}_{\text{tot}} = \hat{v}_m \cdot \hat{v}_n' \frac{1}{4\pi} \sum_n \int_{s_{n+}} \mu_n v_n I_n \frac{1}{R} ds' +=
\hat{v}_m \cdot \hat{v}_n' \frac{1}{4\pi} \sum_n \int_{s_{n+}} (\mu_1 - \mu_2) v_n I_n \frac{1}{R} ds',
\]

and

\[
\psi_{\text{tot}} = \frac{1}{4\pi \varepsilon_0} \sum_n \int_{s_n} u_n \rho_n \frac{1}{R} ds'.
\]

\( \hat{v}_m \) is the tangential unit vector at the match point defined in the direction of the surface and \( \frac{\partial}{\partial s} \) is the tangential derivative w.r.t. the match point. Using cylindrical coordinates, \( R \) can be taken as

\[
R = \sqrt{(\rho \cos \phi - \rho' \cos \phi')^2 + (\rho \sin \phi - \rho' \sin \phi')^2 + (z - z')^2}.
\]

From symmetry we can take \( \phi = \pi/2 \), then \( R \) reduces to

\[
R = \sqrt{(\rho' \cos \phi')^2 + (\rho - \rho' \sin \phi')^2 + (z - z')^2}.
\]

The match points on the dielectric/dielectric interfaces are defined to be in the middle of \( u_n \). The match points on the conductor/dielectric interfaces are defined to be in the middle of the unit pulses \( v_n \), for all surfaces. This is different from the match
point definition in the hybrid method. In the hybrid method the match points on
the conductor/dielectric surfaces in the quasi-static regions are defined to be in the
middle of $u_n$.

5.2. Discrete Version of the Full-Wave Equations

Next, we’ll take a look at the (4.79) with match points in the full-wave region.
Again, the unknowns in the EFIE will be expanded using equations (5.1) and (5.2).
This then gives

$$\bar{E}_1(\bar{r}) = 0$$

(5.9)

where

$$\bar{E}_1(\bar{r}) = \frac{-1}{4\pi} \left[ \hat{z} \cdot \hat{z}' \sum_n j \omega \mu_1 v_n I_n \phi_1 dz' - \hat{z} \cdot \hat{z}' \frac{1}{j \omega \varepsilon_0} \frac{\partial}{\partial z} \sum_n \frac{I_{n+1} - I_n}{\Delta z'} \int_{s_n} v_n \phi_1 dz' \right]$$

$$+ \bar{E}^Q(\rho, z)$$

(5.10)

and

$$\bar{E}^Q(\rho, z) = \hat{z} \cdot \hat{v}'_n \frac{-1}{4\pi} \sum_n \int_{s_n} j \omega \mu_1 v_n I_n \phi_1 dz' - \frac{\partial}{\partial z} \frac{1}{4\pi \varepsilon_0} \int_{s_n} u_n \rho_n \phi_1 dz' \right].$$

(5.11)

Match points in the full-wave region are taken to be on the $z$-axis $\Rightarrow \rho = 0$, thus
reducing $R$ to

$$R = \sqrt{(\rho' \cos \phi')^2 + (\rho' \sin \phi')^2 + (z - z')^2}.$$

(5.12)

5.3. Discrete Version of the Current Continuity Equations

Finally, we’ll look at the discrete version of the current continuity equation. In
(4.82) the unknowns are both currents and charges and they are approximated using
(5.1) and (5.2). This then gives
\[
v_{n+k}I_{n+k}\hat{v}_{n+k} - v_nI_n\hat{v}_n = -j\omega\varepsilon_{ri}\sum_n u_n\rho_nA_n
\]
(5.13)
for free charge and
\[
v_{n+k}I_{n+k}\hat{v}_{n+k} - v_nI_n\hat{v}_n = -j\omega\varepsilon_0(\varepsilon_{ri} - 1)\sum_n u_n\rho_nA_n
\]
(5.14)
for polarized charge. \(A_n\) is the area of the segment the charge pulse is defined on.

Now we have enough discrete integral equations to be evaluated by the MOM.

5.4. New Final Impedance Matrix

The sub matrices to solve the New EFIE can be summarized by the following matrix:

\[
\begin{bmatrix}
\text{Quasi - Static Equations} & \text{Full - Wave Equations} & \text{Current Continuity Equations}
\end{bmatrix}
\begin{bmatrix}
\rho_1 \\
\vdots \\
I_1 \\
\vdots \\
I_{q1}
\end{bmatrix}
= \begin{bmatrix}
0 \\
\vdots \\
E_{inc} \\
\vdots \\
0
\end{bmatrix}
\]

The previous matrix is broken up into three main components as opposed to the four components in the hybrid matrix. The first component consists of the quasi-static equations. This includes the EFIE for match points on the dielectric/dielectric and conductor/dielectric interfaces. The second component consists of the full-wave equations, and the third component consists of the current continuity equations. The unknowns can be solved by evaluating the following matrix:
where \( \rho_n \) is the unknown charge in the quasi-static region, \( I_{qn} \) is the unknown current in the quasi-static region, and \( I_n \) is the unknown current in the full-wave region.

5.5. Evaluating the Derivatives in the New EFIE

The central difference approximation [6] outlined in section 3.7 is used to approximate the derivative in all EFIE. Special care should be taken if the match point is defined on the corner of a conductor/dielectric interface. If this happens then the definition of \( P_{m\pm 1/2} \) can be seen in Figure 15. \( P_{m+1/2} \) is defined at one end of the current pulse wrapping around the corner and \( P_{m-1/2} \) is defined at the other end of the same current pulse. By taking the central difference using these values of \( P_{m\pm 1/2} \), \( P'_m \) approximates the tangential derivative at the match point \( P_m \). \( \Delta s \) is still taken to be \( \Delta s = |P_{m+1/2} - P_{m-1/2}| \).

5.6. Evaluating the Integrals in the New EFIE

Instead of using the \texttt{quad} routine in the Matlab package, a routine was written to implement the Gauss Integration Formula [6]. If we denote the function we wish to integrate from \( a \) to \( b \) as \( f(x) \), then by the \textit{Gauss quadrature formula} we have

\[
\int_{-1}^{1} f(t) dt \approx \sum_{j=1}^{N} \tilde{A}_j f_j. \tag{5.15}
\]

To convert our integral from \([a, b]\) to \([-1, 1]\), we set \( x = \frac{1}{2}[b(t + 1) - a(t - 1)] \) [19].
This then gives $dx = \frac{1}{2} (b - a) dt$. Then the values of $\tilde{A}_j$ and $t_j$ are taken from tables in [20] that calculate the $j^{th}$ zero of the Legendre Polynomial. This then gives

$$\int_a^b f(x) dx = \frac{1}{2} \int_{-1}^1 f(t) dt$$

$$\approx \frac{1}{2} \sum_{j=1}^N \tilde{A}_j f_j$$

$$= \frac{1}{2} [\tilde{A}_1 f(x_1) + \tilde{A}_2 f(x_2) + \tilde{A}_3 f(x_3) + ... \tilde{A}_N f(x_N)]$$

(5.16)

where $x_k = \frac{1}{2} [b(t_k + 1) - a(t_k - 1)]$. This routine greatly reduced the computation time needed for the integration of the EFIE.

5.7. Sources

The expression for the total electric field can be written as $\bar{E}^{tot} = \bar{E}^s_{tan} + \bar{E}^i_{tan} = 0$ where $\bar{E}^s$ is the scattered field and $\bar{E}^i$ is the incident field. The derived EFIE are the expressions representing the scattered field as a result of an incident wave. Now
we need to write a few expressions to represent an incident wave that will excite the problem. The delta source was covered in Section 3.9.1 and will not be covered here. A general expression for the incident field will be covered here.

### 5.7.1. Incident Field

A problem may arise where the currents induced by a plane-wave incident on the antenna are desired. A general expression can be written to represent this incident field as [21]

\[
\vec{E} = [E_\theta \hat{\theta} + E_\phi \hat{\phi}] e^{-jk \vec{r}}
\]  (5.17)

where

\[
\hat{R} = \sin(\theta)\cos(\phi)\hat{x} + \sin(\theta)\sin(\phi)\hat{y} + \cos(\theta)\hat{z},
\]  (5.18)

\[
\hat{\theta} = \cos(\theta)\cos(\phi)\hat{x} + \cos(\theta)\sin(\phi)\hat{y} - \sin(\theta)\hat{z},
\]  (5.19)

\[
\hat{\phi} = -\sin(\phi)\hat{x} + \cos(\phi)\hat{y},
\]  (5.20)

\[
\vec{k} = -k \hat{R},
\]  (5.21)

\[
\vec{r} = x\hat{x} + y\hat{y} + z\hat{z},
\]  (5.22)

and \(k\) is the wave number in the direction of propagation. \(\hat{R}\) is the unit vector indicating the direction of propagation. \(\hat{\theta}\) and \(\hat{\phi}\) are the unit vectors in spherical coordinates indicating the polarization of the incident field. The spherical coordinate system definition can be seen in Appendix E. \(\vec{k}\) is the wave number dotted with the unit vector in the direction of propagation. The position vector \(\vec{r}\) is measured from the origin to the match point on the surface of the antenna.

### 5.7.2. Incident Field Calculations \((\theta_p = 0^\circ)\)

The electric field described by (5.17) was evaluated in Matlab. As a first step, the results for an incident field on a wire calculated by Harrington and Mautz [22] was reproduced here. The problem calculated is shown in Figure 16. It involves

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a thin wire centered on the z-axis and symmetric about the x-y plane. The angles of incidence, $\theta_i$ and $\phi_i$, and polarization angles, $\theta_p$, of the incident electric field are defined in Figures 17 and 18, respectively. The length of the antenna is $l = 2\lambda$ and it has a radius $a$ of $a = \frac{l}{2\pi l/2}$. A source frequency of $f = 500 MHz$ was chosen. This gave a length of $l = 1.2 m$ and a radius $a = 8.08625 mm$. The antenna was divided up into $N = 100$ segments. Figures 19 and 20 show the magnitude and phase of the current induced on the wire as the angle of incident is varied from $0^\circ$ to $90^\circ$ in steps of $15^\circ$. The magnitude of the incident wave was $1V/\lambda$ and the polarization angle is $0^\circ$. These results compare very well with Harrington and Mautz. This indicates that the Matlab code is working correctly for $\theta_p = 0^\circ$. 

Figure 16. Incident Wave on Dipole.
Figure 17. Electric Field Angle of Incidence.

Figure 18. Electric Field Polarization Angle.
Figure 19. Induced Current for $\theta_i = 90^\circ, 75^\circ$, and $60^\circ$; and $\theta_p = 0^\circ$.

Figure 20. Induced Current for $\theta_i = 45^\circ, 30^\circ$, and $15^\circ$; and $\theta_p = 0^\circ$. 
5.7.3. Incident Field Calculations ($\theta_p \neq 0^\circ$)

Then we calculated a problem with $\theta_p \neq 0^\circ$. To do this Mininec Classic [23] was used. Mininec Classic is a commercial software that can simulate the two wires shown in Figure 21. Mininec Classic provides a known result that can be compared to the Matlab calculations. The geometry and segment numbers are the same as the previous problem ($l = 1.2 \text{ m}$, $a = 8.08625 \text{ mm}$, and $N = 100$). The far-field boundary ($r$) is approximately $r \gg \lambda$. Since $\lambda = c/f = .6 \text{ m}$, $r$ was chosen as $r = 100 \text{ m}$. Then a source was placed on the antenna at the origin to provide an electric field magnitude of $\approx 1V/\lambda$ at $y = 100 \text{ m}$. The source antenna was then rotated in the x-z plane to provide an incident field with an angle of polarization w.r.t. the antenna orientation at 100 m. Then, the antenna at 100 m was rotated in the y-z plane to provide an angle of incidence of $75^\circ$. The magnitude and angle of the induced current on the receiving antenna is shown in Figure 22. The two calculations show good results for $\theta_p = 30^\circ$ and $\theta_p = 60^\circ$. The currents presented are for thin-wire antennas only. Further work needs to be done to include problems with quasi-static regions, but initial results look promising.

![Incident Field](image)

Figure 21. Mininec Incident-Wave Problem.
5.8. New EFIE Results

Two problems were chosen to be modeled using the new EFIE. The first problem was the monopole with the capacitive load evaluated by Olsen and Mannikko [3]. This problem was chosen because the hybrid method could be used and experimental results are presented by Olsen and Mannikko. The results from the new EFIE are compared to the experimental results, Mininec Classic, and the Hybrid method. Mininec Classic was chosen because a lumped capacitance could be calculated to approximate the quasi-static region.

The second problem was a monopole with an inductive load. The quasi-static region has a dielectric region that resembles the geometry of a ferrite bead. This problem was chosen because the quasi-static region is inductively dominant for dielectrics with relative permeability other than unity. The results from the new EFIE are compared with the results provided by Mininec Classic and Ansoft [24]. Mininec
Classic was used because a lumped element can be calculated to approximate the inductance of the quasi-static region. Ansoft was chosen for two reasons. First, Ansoft provides results using the Finite Element Method [25] which is based on completely different principles than presented here. Secondly, work done by Kennedy, Long, and Williams [26] suggest that problems similar to this one can be calculated using Ansoft.

5.8.1. Monopole with a Capacitive Load

The input impedance was evaluated for the monopole shown in Figure 23. The figure of the monopole is illustrated here for convenience, the original drawing can be found in [3]. Notice that the step radius is not modeled here. \( \varepsilon_r = 2.32 \) was chosen

![Figure 23. Monopole Antenna with a Capacitive Load.](image-url)
for the dielectric between the capacitor plates and $\mu_r = \mu_0$. The source frequency
was then varied to preserve the quasi-static characteristics of the capacitive region.
In particular the frequency of the source was chosen to vary from $1.8GHz \leq f \leq
2.3GHz$. This corresponded to the admittance plots found in [17].

The problem was first evaluated in Matlab using the new EFIE. The total
number of segments was $N=230$, which resulted in 409 unknowns. This number
was determined by increasing the number of unknowns until the input impedance
converged. This problem was then evaluated using the Hybrid Code. The total
number of segments was $N=210$, which resulted in 393 unknowns. As a final check
Mininec Classic was used with a lumped element of $0.447pF$ placed $20.7144$ mm
above the ground plane. This value of capacitance was suggested by Olsen and Mannikko.
The calculated input conductance and susceptance for the previous three methods
are shown in Figures 24 and 25, respectively. The measured values in Figures 24 and
25 are approximated from the results presented in [3]. The input admittance of a
monopole without a quasi-static region is also shown to illustrate how the capacitive
quasi-static region affects the input admittance. The input admittances calculated
using the new EFIE show a very good correlation with the hybrid results and the
measured results. This helps to ensure that the values calculated using the new EFIE
are valid.

Another calculation was performed for a broader range of frequencies. The input
reactance of the capacitively loaded monopole is shown in Figure 26 for $1.0GHz \leq
f \leq 2.5GHz$. The results of the monopole without a capacitive load are also presented
in Figure 26 to illustrate the effect of the quasi-static regions. Figure 26 illustrates
that the new EFIE calculations correspond very well with the results from the hybrid
method. The results from this problem show that the new EFIE can effectively
calculate the input impedances of a monopole with capacitive loading.
Figure 24. Conductance of Monopole Antenna with a Capacitive Load.

Figure 25. Susceptance of Monopole Antenna with a Capacitive Load.

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5.8.2. Monopole with an Inductive Load

The next antenna is the inductively loaded monopole shown in Figure 27. The quasi-static region is made up of a dielectric region wrapping around a wire. This is equivalent to placing a ferrite bead around a wire to introduce some inductance. The dielectric has permittivity $\varepsilon_{r_1} = 1.01$ and permeability $\mu_{r_1} = 30$. The values of $a$ and $b$ are $0.15\ mm$ and $2\ mm$, respectively (inner and outer diameter of the bead).

The monopole in Figure 27 was evaluated in Matlab using the new EFIE. The number of segments was increased until the input impedance converged, which happened when $N$ was 160. To validate the results from the Matlab code, the quasi-static region was approximated by an equivalent lumped inductor element and entered into Mininec Classic. To calculate the equivalent element, the problem in Figure 28 was considered. The inductance of the bead can be approximated as

Figure 26. Reactance of Monopole Antenna with a Capacitive Load.
where \( N_t \) is the number of turns by the wire. Assuming that the magnetic field in the bead is

\[
\vec{H} = \frac{I}{2\pi \rho_l} \hat{a}_\phi
\]

(5.24)

the inductance can be approximated as

\[
L \approx \int_0^l \int_a^b \frac{\mu_r \mu_0 L}{I} d\rho dz \hat{a}_\phi \cdot \hat{a}_\phi = \int_0^l \int_a^b \frac{\mu_r \mu_0}{2\pi \rho_l} d\rho dz = \frac{\mu_r \mu_0}{2\pi} l l_n \left( \frac{b}{a} \right) l.
\]

(5.25)

For a bead with \( \mu_r = 30 \) the dimensions give \( L = 15.542nH \). The lumped inductance
was then defined at a node in Mininec Classic that corresponded to the center of the quasi-static region in Figure 27. Ansoft was also chosen to validate the results from the new EFIE. The geometry entered into Ansoft is described in Appendix F. It should be noted here that the surfaces defined in Ansoft were not smooth. Every surface was defined to be a tetrahedron with 14 sides. The problem was too large for the PC to provide valid results using smooth surfaces. The frequency was then swept from 0.3GHz to 1.8GHz. This ensured that the quasi-static assumptions were still valid. The calculated the input resistance and reactance for this sweep are shown in Figures 29 and 30, respectively. The input impedance of a monopole without the ferrite bead is also presented in Figures 29 and 30. Again, this is to illustrate the effects of the quasi-static region on the input impedances.
Figure 29. Input Resistance for Inductively Loaded Monopole.

Figure 30. Input Reactance for Inductively Loaded Monopole.
Figures 29 and 30 show that there is excellent correlation between the results of the new EFIE, Mininec Classic, and Ansoft. This helps ensure that the results provided by the new EFIE are valid for problems with inductive quasi-static regions, especially because Ansoft is based entirely on a different set of principles.

The input reactance was also calculated for frequencies near the resonant point. The results from this sweep are presented in Figure 31. The monopole without a quasi-static region is also shown. Again, this helps illustrate the effects of the quasi-static region on the input reactance. Figure 31 shows that the calculations by the new EFIE are very similar to the Mininec Classic and Ansoft Calculations. This also builds the confidence in the new EFIE results. A similar correlation exists between the new EFIE results, Mininec Classic, and Ansoft for the input resistance.

A small summary of the Matlab code for all the calculations presented in

![Figure 31. Input Reactance Near Resonant Point.](image-url)
Chapter 5 is included in Appendix G. The code has several subroutines that generate the source and match points used in the new EFIE calculations and that evaluate the integrals numerically in all the new EFIE. To manage all these subroutines, a Graphical User Interface (GUI) was written using the Matlab GUI editor. The name QUICNEC [27] has been given to this Matlab GUI. QUICNEC stands for Quasi-Static Inductive Capacitive Numerical Electromagnetics Code. The GUI allows the user to easily input an array of axially symmetric problems. More information on the capabilities and limitations of QUICNEC can be found in [27].

5.9. Design Suggestions

In this section, several design suggestions are presented. Most of the suggestions come from using the hybrid code and QUICNEC. In the quasi-static region, it is assumed that

\[ k_i D \ll 1 \]  

(5.26)

where, again, D is the overall dimension of the quasi-static region. Equation (5.26) can also be written in the following manner:

\[ \omega D \sqrt{\varepsilon_0 \mu_0 \varepsilon_r \mu_r} \ll 1. \]  

(5.27)

Equation (5.27) helps illustrate the effect that permittivity and permeability has on the inequality. If a value of \( 1/10 \ll 1 \) was chosen such that \( \omega D \sqrt{\varepsilon_0 \mu_0 \varepsilon_r \mu_r} \leq 1/10 \) was a guideline for the design of a quasi-static region, then D could be written in terms of the material properties of the quasi-static region in the following manner:

\[ D \leq \frac{\alpha \lambda}{\sqrt{\varepsilon_r \mu_r}} \]  

(5.28)

where \( \alpha = \frac{1}{20\pi c \sqrt{\varepsilon_0 \mu_0}} \). Figure 32 shows, for several frequencies, the recommended overall dimension of the quasi-static region vs. the product of \( \varepsilon_r \) and \( \mu_r \). Some
results have been valid for values of D above the curves, but this plot should provide a good design tool.

The quasi-static surfaces have been divided into 10 segments each for all the problems presented in the previous sections. Increasing this number to 15 or 20 significantly increased the computation time but with little change in calculated values. The number of full-wave segments corresponded very closely to the number of segments used in Mininec Classic. The number of full-wave segments used in the hybrid code and QUICNEC was approximately the same number that provided convergence in Mininec Classic when no lumped elements were used.

These suggestions hopefully will provide guidance in choosing the number of segments to use in QUICNEC. The final guideline is to increase or decrease the number of segments until the selected parameter (e.g., input impedance) converges.
An integral equation technique has been successfully developed to compute the fields from axially symmetric antenna problems with quasi-static and full-wave regions. For the case of a monopole with a capacitive load the input impedance calculated using the new integral equations compared very well to those obtained via the hybrid method and measured results. This shows that the new integral equations can calculate detailed information about capacitively dominant quasi-static regions. Also, the input impedance calculated using the new integral equations, Mininec Classic, and Ansoft compared very well for the case of a monopole with an inductive load (i.e. ferrite bead). These results show that the new integral equations provide very detailed results for quasi-static regions with significant inductive effects. Since Ansoft compared very well to the results provided using the new EFIE the user can feel confident in the new EFIE presented here.

Further work could be done to write code that includes quasi-static and full-wave regions with arbitrary shape. This would allow the new integrals to be used in computing detailed results for circuit board designs and arbitrary radiating environments. Experimental results could also be obtained to validate the simulated results. Another interesting topic would be to determine what constraints are needed to ensure that results obtained using Mininec with lumped elements are accurate. A derivation of the EFIE for problems with finite conductivity ($0 < \sigma < \infty$) could be also be performed. Deriving time domain versions of the EFIE is another interesting project.


APPENDIX A. CYLINDRICAL COORDINATE SYSTEM

The cylindrical coordinate system defined throughout the thesis can be seen in Figure 33. The point \( P \) is described by the vector

\[
\mathbf{P} = P_\rho \hat{a}_\rho + P_\phi \hat{a}_\phi + P_z \hat{a}_z
\]

where \( \hat{a}_\rho, \hat{a}_\phi, \) and \( \hat{a}_z \) are unit vectors in the \( \rho, \phi, \) and \( z \) directions respectively. Line integration \( d\mathbf{L} \) is described by

\[
d\mathbf{L} = d\rho \hat{a}_\rho + \rho d\phi \hat{a}_\phi + dz \hat{a}_z
\]

and surface integration \( d\mathbf{S} \) is described by

\[
d\mathbf{S} = \rho d\phi dz (\pm \hat{a}_\rho)
\]

\[
= d\rho dz (\pm \hat{a}_\phi)
\]

\[
= \rho d\rho d\phi (\pm \hat{a}_z).
\]

Figure 33. Cylindrical Coordinate System.
APPENDIX B. HYBRID MATLAB CODE

The file displayed here is

`monopole_dielectric_example.m`

The flow chart that describes these Matlab calculations can be seen in Figure 34. The file

`monopole_dielectric_example.m`

contains all the problem information and calls the following m-files to evaluate the appropriate EFIE. The file

`full_wave_integration_routines.m`

calculates all the EFIE with match points in the full-wave region and the full-wave contribution on the dielectric/dielectric surfaces in the quasi-static regions. The file

`quasi_integration_routines.m`

calculates all the EFIE with match points in the quasi-static region except for the full-wave contribution on the dielectric/dielectric.

The current continuity equations and the matrix inversion are evaluated in

`monopole_dielectric_example.m`

The derivatives are evaluated throughout all the m-files. The top sub-block in Figure 34 is shown in Figure 35 with more detail. The first part of the file globalizes the variables to be used in all three subroutines. Then the source and match points are generated to be used in the hybrid EFIE. The next subroutines call the appropriate functions to evaluate the hybrid EFIE. Then the matrix is inverted and input impedance is calculated.
Figure 34. Hybrid Matlab Flowchart.

Figure 35. Hybrid Main Sub-Block.
% Author: Ben Braaten (3/5/2005)
% Hybrid code for monopole-dielectric example in Section 3.10.
% This code was used to compute the EFIE in the hybrid method.
% Written in Matlab.

%cyan
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% The following lines of code calculate the impedance
% matrix using the point matching technique. The observation point
% is at point m and the inner product is integrated
% over delta z and/or delta phi (depending on the surface
% of the source). The two functions are called
% 'quasi_integration_routines' and 'full_wave_integration_routines'.
% These two routines are called when the matchpoints are in
% the quasi-static and full-wave regions respectively.
% The impedance matrix 'Z_final' is then filled in
% with these routines. The voltage matrix 'V' is then
% defined and a delta source
% used to represent the incident field.
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

clc clear all disp('START')

%global z_full amq_diel_diel zmq_diel_diel s del_zm_full m_charge
%global n_charge del_z2 del_z3 zpq_mininec zmq_mininec zp_mininec
%global zm_mininec m_current n_current v_quasi_current v_quasi
%global v_full eps_r eps_l rho_m z_m zp_full zm_full Z Z_final
%global a w tolerance N1 del_z1 lambda u0 e0 m_charge
%global n_charge z_p a_p z_p_current

z_0=0; %various values for z along the antenna.
z_1=0.0547;
z_2=0.05683;
z_3=0.0611;
s=0; %counter.
L=0.0547; %length of monopole in meters.
a=0.075e-3; %radius of antenna in meters.
f=2e9; %frequency
epsilon_0=8.854e-12; %Defines constants.
u_0=1.256e-6; %omega.
c=3e8; %speed of propagation in free space.
lambda=c/f; %wave length.
c_1=-1/(4*pi*i*w*epsilon_0); %calculated constant using E=jwA+delPHI.
k=2*pi/lambda; %wave number.
tolerance=1e-6; %integration tolerance.
Z=zeros(17,17); %size of impedance calculation matrix.
Z_final=zeros(17,17); %size of final impedance matrix.
eps_l=3; %Relative epsilon to the%left of the diel./dielectr boundary.
eps_r=1; %Relative epsilon to the%right of the diel.diel. boundary.
fill=1; %function 'filler'
N1=3; %number of full wave segments.
N2=1; %number of segments along vertical%diel./cond. interface.
N3=2; %number of segments along horizontal%diel./cond. interface.
N4=2; %number of segments along vertical

x=0;
x_f=0.0547;
x_2=0.05683;
x_3=0.0611;
eps_r=0; %counter.
L=0.0547; %length of monopole in meters.
a=0.075e-3; %radius of antenna in meters.
f=2e9; %frequency
epsilon_0=8.854e-12; %Defines constants.
u_0=1.256e-6; %omega.
c=3e8; %speed of propagation in free space.
lambda=c/f; %wave length.
c_1=-1/(4*pi*i*w*epsilon_0); %calculated constant using E=jwA+delPHI.
k=2*pi/lambda; %wave number.
tolerance=1e-6; %integration tolerance.
Z=zeros(17,17); %size of impedance calculation matrix.
Z_final=zeros(17,17); %size of final impedance matrix.
eps_l=3; %Relative epsilon to the%left of the diel./dielectr boundary.
eps_r=1; %Relative epsilon to the%right of the diel.diel. boundary.
fill=1; %function 'filler'
N1=3; %number of full wave segments.
N2=1; %number of segments along vertical%diel./cond. interface.
N3=2; %number of segments along horizontal%diel./cond. interface.
N4=2; %number of segments along vertical

Tious values for z along the antenna.
N5=2;

del_z1=x_1/N1;
del_z2=x_2-x_1;
del_z3=x_3/N3;
del_z4=(x_3-x_2)/N4;
del_z5=x_5/N5;

del_zm_full=[del_z1*[ones(1,N1-1) del_z2/2]/2];
del_zm_diel_diel=[del_z4/10,del_z4/10, del_z5/10,del_z5/10];

v_quasi=[0 0 1; 0 -1 0; 0 -1 0; 0 0 1; 0 0 1; 0 -1 0; 0 -1 0];
v_quasi_normal=[0 1 0; 0 0 1; 0 1 0];
v_full=[0 1 0; 0 1 0; 0 0 1];

a=[a-a del_x3/2, del_x3/2, a, a, a-del_x3/2, del_x3/2];
a_p=[a, a-a-del_x3, a-del_x3, 0; a, a; a, a; a-del_x3, a-del_x3, 0];

z_p=[x_1 x_2; x_2 x_2; x_2 x_2; x_2 x_3-del_x4; x_3-del_x4 x_3-del_x4; x_3-del_x4 x_3-del_x4; x_3-del_x4 x_3-del_x4; x_3-del_x4 x_3-del_x4];

zp_full=[del_z1/2:del_z1:x_1-del_z1/2 del_x2/2];

zm_minec=[zm_full',zm_full_plus_half',zm_full_minus_half',zm_full_plus_half',zm_full_minus_half'];

zp_minec=zp_full;
flag=1; disp('CALCULATING THE FULL-WAVE/FULL-WAVE CONTRIBUTION')
for n=1:N1
  s=1;
  for n=1:N1
    if m and n+1/2 to n-1/2
      Z_non_gradient=full_wave_integration_routines(flag,s,m,n,fill);
      Z_non_gradient=Z_non_gradient*(del_zm_full(m))*k^2;
      s=s+1;
    end
    if m+1/2 and n to n+1
      Z_gradient_a_plus=full_wave_integration_routines(flag,s,m,n,fill);s=s+1;
    end
    if m-1/2 and n to n+1
      Z_gradient_a_minus=full_wave_integration_routines(flag,s,m,n,fill);s=s+1;
    end
    if m+1/2 and n-1 to n
      Z_gradient_b_plus=full_wave_integration_routines(flag,s,m,n,fill);s=s+1;
    end
    if m-1/2 and n-1 to n
      Z_gradient_b_minus=full_wave_integration_routines(flag,s,m,n,fill);s=s+1;
    end
    Z_final(m_full_current(m),n_full_current(n))=(Z_non_gradient-...
      (Z_gradient_a_plus-Z_gradient_a_minus)/(del_zm_full(1))-...
      (Z_gradient_b_plus-Z_gradient_b_minus)/(del_zm_full(1)));
  end
end

% **********************************************************
% The following for-loops calculate the CHARGE contribution
% of the quasi-static regions to the LOWER full wave region.
% **********************************************************
flag=2; disp('CALCULATING THE CHARGE CONTRIBUTION OF THE QUASI-STATIC REGION ON THE FULL-WAVE REGION')
for m=1:N1
  s=1;
  for n=1:sum([N2 N3 N4 N5])
    Z_final(m_charge(m),n_charge(n))=(full_wave_integration_routines(flag,s,m,n,fill));
  end
end

% **********************************************************
% The following for-loops calculate the CURRENT contribution of the quasi-static regions to the
% LOWER full wave region.
% **********************************************************
flag=3; disp('CALCULATING THE CURRENT CONTRIBUTION OF THE QUASI-STATIC REGION ON THE FULL-WAVE REGION')
for m=1:N1;
  for n=1:sum([N2 N3 N4 N5])-1
    Z_final(m_current(m),n_current(n))=(full_wave_integration_routines(flag,s,m,n,fill));
  end
end

% **********************************************************
% The following for-loops calculate the CHARGE contribution of the quasi-static regions
% on itself.
% **********************************************************
% for match points on the vertical conductor/dielectric interface and horizontal
% conductor/dielectric interfaces.
CALCULATING THE CHARGE CONTRIBUTION ON ITSELF IN THE QUASI-STATIC REGION

flag=1;
for n=1:1:3;
  m_charge=m;
  for m=1:1:3;
    n_charge=n;
    Z_final(m_charge,n_charge)=(quasi_integration_routines(flag,m,a_p(n,:),z_p(n,:),fill));
end
end

% for match points on the vertical dielectric/dielectric interface and horizontal
% dielectric/dielectric interfaces.
flag=2; for n=4:1:7;
  m_charge=m;
  for m=1:1:7;
    n_charge=n;
    normal=v_quasi(n,3);
    Z_final(m_charge,n_charge)=(quasi_integration_routines(flag,m,a_p(n,:),z_p(n,:),normal));
end
end

If the source point and match point are on the same segment then equation 2.8 on page 21
of the hybrid thesis is solved for the match charge by subtracting both sides by
eps_l+eps_r)/(2*eps0).
if m==n;
  Z_final(m_charge,n_charge)=Z_final(m_charge,n_charge)-(eps_l+eps_r)/(2*eps0);
else
end
end
end

%******************************************************************************
% The following function call calculates the contribution of the current
% in the fullwave region on the conductor/dielectric interface.
%******************************************************************************
flag=3;
disp('CALCULATING THE FULL-WAVE CURRENT EFFECTS ON THE COND./DIEL.')
disp('INTERFACE OF THE QUASI-STATIC REGION.')
quasi_integration_routines(flag,s,fill,fill,fill,fill,fill)
flag=0;
Z_final(1:3,7+1:N1)=Z_final(1:3,8+1:N1)-Z_final(1:3,7+1:N1)/del_z1;
Z_final(1:3,17)=1;

%******************************************************************************
% The following for-loops calculate the contribution of the
% full wave region on the vertical and horizontal diel./diele. interface.
%******************************************************************************
disp('CALCULATING THE FULL-WAVE CURRENT EFFECTS ON THE DIEL./DIEL. INTERFACE')
disp('OF THE QUASI-STATIC REGION.')
flag=4;
for m=1:4
  s=1;
  for n=1:3

for m and n+1/2 to n-1/2
Z_non_gradient=full_wave_integration_routines(flag,s,m,n,fill);
Z_nongradient=Z_non_gradient;
s=s+1;
for m+1/2 and n to n+1
Z_gradient_a_plus=full_wave_integration_routines(flag,s,m,n,fill);
s=s+1;
for m-1/2 and n to n
Z_gradient_a_minus=full_wave_integration_routines(flag,s,m,n,fill);
s=s+1;
for m+1/2 and n-1 to n
Z_gradient_b_plus=full_wave_integration_routines(flag,s,m,n,fill);
s=s+1;
for m-1/2 and n-1 to n
Z_gradient_b_minus=full_wave_integration_routines(flag,s,m,n,fill);
dot_product=dot(v_full(n,:),v_quasi_normal(m,:));
Z_final(m_diel_current(m),n_full_current(n))=(dot_product*...
Z_nongradient*k^2-(((Z_gradient_a_plus-Z_gradient_a_minus)/(del_zm_diel_diel(1,m)...)...-
(Z_gradient_b_plus-Z_gradient_b_minus)/(del_zm_diel_diel(1,m)...)...)/
(del_zm_full(1,1))))*c1*(eps_r-eps_l);
s=1;
end
end
disp('CALCULATING QUASI-STATIC CURRENTS')
Vzeros(17,1);
B=-1/(-c1); V(8,1)=E;
Z_final(11,1)=j*w*2*pi*a*del_z2;
Z_final(11,10)=1;
Z_final(11,11)=1;
Z_final(12,2)=3*(-j*w*pi*a^2+j*w*pi*del_z3^2);
Z_final(12,12)=3;
Z_final(12,11)=1;
Z_final(12,11)=1;
Z_final(13,2)=2*(j*w*pi*a^2-j*w*pi*del_z3^2);
Z_final(13,13)=2;
Z_final(13,12)=2;
Z_final(13,12)=2;
Z_final(14,4)=3*(-j*w*pi*a^2-j*w*pi*del_z3^2);
Z_final(14,14)=1;
Z_final(14,13)=1;
Z_final(14,13)=1;
Z_final(15,5)=3*(-j*w*pi*a^2-j*w*pi*del_z3^2);
Z_final(15,15)=1;
Z_final(15,14)=1;
Z_final(15,14)=1;
Z_final(16,6)=3*(-j*w*pi*a^2+j*w*pi*del_z3^2);
Z_final(16,16)=1;
Z_final(16,15)=1;
Z_final(16,15)=1;
Z_final(17,7)=3*(-j*w*pi*a^2+j*w*pi*del_z3^2);
Z_final(17,17)=1;
Z_final(17,16)=1;
Z_final(17,16)=1;
Z_final(18,8)=3*(-j*w*pi*a^2-j*w*pi*del_z3^2);
Z_final(18,18)=1;
Z_final(18,17)=1;
Z_final(18,17)=1;
Z_final(19,9)=j*w*pi*a*del_z2;
Z_final(19,19)=1;
Z_final(19,18)=1;
Z_final(19,18)=1;
Z_final(20,10)=j*w*pi*a*del_z2;
Z_final(20,20)=1;
Z_final(20,19)=1;
Z_final(20,19)=1;
Z_final(21,11)=j*w*pi*a*del_z2;
Z_final(21,21)=1;
Z_final(21,20)=1;
Z_final(21,20)=1;
Z_final(22,12)=j*w*pi*a*del_z2;
Z_final(22,22)=1;
Z_final(22,21)=1;
Z_final(22,21)=1;
Z_final(23,13)=j*w*pi*a*del_z2;
Z_final(23,23)=1;
Z_final(23,22)=1;
Z_final(23,22)=1;
Z_final(24,14)=j*w*pi*a*del_z2;
Z_final(24,24)=1;
Z_final(24,23)=1;
Z_final(24,23)=1;
Z_final(25,15)=j*w*pi*a*del_z2;
Z_final(25,25)=1;
Z_final(25,24)=1;
Z_final(25,24)=1;
Z_final(26,16)=j*w*pi*a*del_z2;
Z_final(26,26)=1;
Z_final(26,25)=1;
Z_final(26,25)=1;
Z_final(27,17)=j*w*pi*a*del_z2;
Z_final(27,27)=1;
Z_final(27,26)=1;
Z_final(27,26)=1;
Z_final(28,18)=j*w*pi*a*del_z2;
Z_final(28,28)=1;
Z_final(28,27)=1;
Z_final(28,27)=1;
Z_final(29,19)=j*w*pi*a*del_z2;
Z_final(29,29)=1;
Z_final(29,28)=1;
Z_final(29,28)=1;
Z_final(30,20)=j*w*pi*a*del_z2;
Z_final(30,30)=1;
Z_final(30,29)=1;
Z_final(30,29)=1;
Z_final(31,21)=j*w*pi*a*del_z2;
Z_final(31,31)=1;
Z_final(31,30)=1;
Z_final(31,30)=1;
inv(Z_final)*V
Z_in=1/inv(Z_final(8,1))
disp('Done')
APPENDIX C. FULL-WAVE INTEGRATION ROUTINE

The file displayed here is

full_wave_integration_routines.m.

The bottom left sub-block in Figure 34 is shown in Figure 36 with more detail. The first part of the file globalizes the variables to be used in all three subroutines. Then the appropriate subroutines are used to evaluate the needed EFIE. Integration is performed by Gaussian Quadrature, quad, and dblquad.

![Figure 36. Hybrid Full-Wave Integration Sub-Block.](image)
function [Z_calc]=full_wave_integration_routines(flag,s,m,n,fill)

% The following lines of code calculate the impedance matrix using the point matching technique. The observation point is at point m and the inner product is integrated over delta z.

%/constants calculated for the full-wave routine.
cl=1/(2*pi*j*w*e0);
N1=1.38629436112;
eps_diel=0.09663343259;
eps_diel_min=0.03900292383;
eps_diel_min=0.03742563713;
eps_diel_min=0.01451196212;

w=5;
a0=1248599397;
a1=0.0585344576;
a2=0.0333255346;
a3=0.0044787012;

if flag==1;

% The following routine calculates the contribution of the full-wave region on itself. The loops calculate the integral using the 'rectangle' approximation for the integration and a central difference approximation for the derivative.

segnum=1000001;

z=(zp_mininec(n,s):segnum:zp_mininec(n+1,s));
I3_int=(exp(-j/k*sqrt((zm_mininec(m,s)-z).^2+a.^2))-1)./sqrt((zm_mininec(m,s)-z).^2+a.^2);
I3=sum(I3_int)/segnum;

I2_ellip_int=(1/(pi*a))*(2*a./sqrt((zm_mininec(m,s)-z).^2+4*a.^2)).*(a0+a1*((zm_mininec(m,s)-z).^2./((zm_mininec(m,s)-z).^2+4*a.^2))+a2*((zm_mininec(m,s)-z).^2./((zm_mininec(m,s)-z).^2+4*a.^2)).^2+a3*((zm_mininec(m,s)-z).^2./((zm_mininec(m,s)-z).^2+4*a.^2)).^3+b0+b1*((zm_mininec(m,s)-z).^2./((zm_mininec(m,s)-z).^2+4*a.^2))+b2*(zm_mininec(m,s)-z).^2./((zm_mininec(m,s)-z).^2+4*a.^2))+(b3*(zm_mininec(m,s)-z).^2./((zm_mininec(m,s)-z).^2+4*a.^2)).^3.*...
\[
\log\left(\frac{1}{\left|z_{min} - z\right|^2 + 4a^2}\right);
\]

\[
I_2_{\text{ellip}} = \frac{1}{\pi a} \log\left|\frac{z_{min} - z}{8a}\right|
\]

\[
I_2_{\text{sing}} = \frac{1}{\pi a} \log\left|\frac{z_{min} - z}{8a}\right|
\]

\[
I_2 = I_2_{\text{ellip}} + I_2_{\text{sing}}
\]

\[
I_1 = \frac{1}{\pi a} \left(\left(z_{min} - z_{p_{min}}(n+1,s)\right) \times \log\left|\frac{z_{min} - z_{p_{min}}(n+1,s)}{8a}\right| - 1\right)
\]

\[
Z_{\text{calc}} = (I_1 + I_2 + I_3)
\]

elseif flag == 2;

The following routine calculates the contribution the charge in the quasi-static region has on the full-wave region by evaluating the integral at \(m+1/2\) and \(m-1/2\) and then uses the difference approximation to evaluate the gradient. The integral is a double integral over just the source charge pulse.

if dot(v_full(m,:),v_quasi(n,:)) == 1;

\[
s = 2;
Z_{\text{plus half}} = a * j * w * \text{dblquad}(\text{inline}(\left['\exp(-j*sqrt(', num2str(k), ')*sqrt(('), num2str(a), '*cos(phi)).^2+', num2str(a), '*sin(phi)).^2+', num2str(zm_mininec(m,s)), ').^2')) / sqrt(((', num2str(a), '*cos(phi)).^2+', num2str(a), '*sin(phi)).^2+', num2str(zm_mininec(m,s)), ').^2))', 0, 2*pi, z_p(n,1), z_p(n,2), tolerance);
\]

\[
s = s + 1;
Z_{\text{minus half}} = a * j * w * \text{dblquad}(\text{inline}(\left['\exp(-j*sqrt(', num2str(k), ')*sqrt(('), num2str(a), '*cos(phi)).^2+', num2str(a), '*sin(phi)).^2+', num2str(zm_mininec(m,s)), ').^2')) / sqrt(((', num2str(a), '*cos(phi)).^2+', num2str(a), '*sin(phi)).^2+', num2str(zm_mininec(m,s)), ').^2))', 0, 2*pi, z_p(n,1), z_p(n,2), tolerance);
\]

\[
Z_{\text{calc}} = (Z_{\text{plus half}} - Z_{\text{minus half}})
\]

elseif dot(v_full(m,:),v_quasi(n,:)) == 0;

\[
s = 2;
Z_{\text{plus half}} = a * j * w * \text{dblquad}(\text{inline}(\left['\exp(-j*sqrt(', num2str(k), ')*sqrt(('), num2str(a), '*cos(phi)).^2+', num2str(a), '*sin(phi)).^2+', num2str(z_p(n,1)), '-', num2str(zm_mininec(m,s)), ').^2')) / sqrt(((', num2str(a), '*cos(phi)).^2+', num2str(a), '*sin(phi)).^2+', num2str(z_p(n,1)), '-', num2str(zm_mininec(m,s)), ').^2))', a_p(n,2), a_p(n,1), 0, 2*pi, tolerance);
\]

\[
s = s + 1;
Z_{\text{minus half}} = a * j * w * \text{dblquad}(\text{inline}(\left['\exp(-j*sqrt(', num2str(k), ')*sqrt(('), num2str(a), '*cos(phi)).^2+', num2str(a), '*sin(phi)).^2+', num2str(z_p(n,1)), '-', num2str(zm_mininec(m,s)), ').^2')) / sqrt(((', num2str(a), '*cos(phi)).^2+', num2str(a), '*sin(phi)).^2+', num2str(z_p(n,1)), '-', num2str(zm_mininec(m,s)), ').^2))', a_p(n,2), a_p(n,1), 0, 2*pi, tolerance);
\]

\[
Z_{\text{calc}} = (Z_{\text{plus half}} - Z_{\text{minus half}})
\]
else
    Do nothing
end

elseif flag==3;

%************************************************************************************
%The following routine calculates the contribution the current in the quasi-static region
%has on the full-wave region.
%************************************************************************************

if dot(v_full(m,:),v_quasi_current(n,:))==1;
    g=dot(v_full(m,:),v_quasi_current(n,:));
    Z_calc=g*(del_zm_full(m))*(w*w*e0*u0/(1))*quad(inline(...
        ['(exp(-j*(2*pi/.15).*sqrt(',num2str(a),'.*cos(pi/2)).^2+(',...num2str(a),'.*sin(pi/2)).^2+(z-',num2str(zm_full(m)),').^2)))./(sqrt(',num2str(a),'.*cos(pi/2)).^2+(',num2str(a),'.*sin(pi/2)).^2+(z-',num2str(zm_full(m)),').^2))'
    ));
else
    g=dot(v_full(m,:),v_quasi_current(n,:));
    Z_calc=g;
end

elseif flag==4;

%************************************************************************************
%The following routine calculates the contribution of the current in
%the full-wave region on the diel./diel interfaces of the quasi-static region.
The loops calculate the integral using the 'rectangle' approximation for integration and a difference approximation for the derivative. Pulse functions are used as expansion and weighting functions.
%************************************************************************************

segnum=1000001;
zm=zmq_diel_diel(m,s); %match point for z.
am=amq_diel_diel(m,s); %match point for rho.
zm_lower=zpq_mininec(n,s); %lower limit of source point
zm_upper=zpq_mininec(n+1,s); %upper limit of source point
z=[zm_lower:1/segnum:zm_upper]; %integration points
d=(zm-z).^2+am^2;
I3_int=((exp(-j*k*sqrt(d+a^2)))./sqrt(d+a^2));
I3=sum([I3_int(1,2:length(z)-1) .5*I3_int(1,1) .5*I3_int(1,length(z))])/segnum;
Z_calc=I3;
else
end
APPENDIX D. QUASI-STATIC INTEGRATION ROUTINE

The file displayed here is

\texttt{quasi\_integration\_routines.m}.

The bottom right sub-block in Figure 34 is shown in Figure 37 with more detail. The first part of the file globalizes the variables to be used in all three subroutines. Then the appropriate subroutines are used to evaluate the needed EFIE. Integration is performed by rectangular approximation and \texttt{quad}.

\begin{figure}[h]
\centering
\begin{tabular}{|c|}
\hline
\texttt{quasi\_integration\_routines.m} \\
Sets up Gaussian integration variables \\
\begin{tabular}{|c|}
\hline
Integration routine for the quasi-static charge contribution on the quasi-static region on the conductor/dielectric interfaces \\
\hline
Integration routine for the quasi-static charge contribution on the quasi-static region on the dielectric/dielectric interfaces \\
\hline
Integration routine for the full-wave effects on the conductor/dielectric interfaces in the quasi-static region \\
\hline
\end{tabular}
\end{tabular}
\caption{Hybrid Quasi-Static Integration Sub-Block.}
\end{figure}
function \[Z_{int}\] = quasi_integration_routines(flag, s, a_p, z_p, normal)

```matlab
clear A clear Z_int

global amq_diel_diel zmq_diel_diel rho_m z_m eps_r
global eps_l N1 w e0 tolerance z_full Z

seg_num = 101; % this is the number the integration w.r.t. phi_p is broken up into.

phi_p = 0:2*pi/seg_num:2*pi; % values of phi for the source points.
e0 = 8.854e-12; % defines Constants
phi_m = pi/2; % value of phi at the match points.

if flag == 1
    % This 'if' statements calculates the charge contributions in the quasi-static regions to it's self on the conductor/
    % dielectric interfaces.
    %*********************************************************
    %Then the BOTH the match point and source point are on a vertical axis.
    if (a_p(1)-a_p(2))==0
        p = 0;
        A = (2*pi/seg_num)*(abs(z_p(2)-z_p(1))/seg_num); % area of rectangle
        for z = z_p(1):(z_p(2)-z_p(1))/seg_num:z_p(2);
            p = p + 1;
            Zq(p,:) = a_p(1)/(4*pi*e0)*1./((sqrt((rho_m(s)*cos(phi_m)-a_p(1).*cos(phi_p)).^2 ... + (rho_m(s)*sin(phi_m)-a_p(1).*sin(phi_p)).^2 + (z-z_m(s)).^2)));
        end
    
    %Then the match point and source point are on a DIFFERENT axis.
    else
        p = 0;
        A = (2*pi/seg_num)*(abs(a_p(2)-a_p(1))/seg_num); % area of rectangle
        for a_p = a_p(1):(a_p(2)-a_p(1))/seg_num:a_p(2);
            p = p + 1;
            Zq(p,:) = a_p./(4*pi*e0).*1./((sqrt((rho_m(s)*cos(phi_m)-a_p.*cos(phi_p)).^2 ... + (rho_m(s)*sin(phi_m)-a_p.*sin(phi_p)).^2 + (z_p(1)-z_m(s)).^2)));
        end
    end
    Z_int = sum(sum(Zq))*A;

elseif flag == 2
    %*********************************************************
    % This 'if' statements calculates the charge contributions in the quasi-static regions to itself on the dielectric/
    % dielectric interfaces.
    %*********************************************************
    %Then the BOTH the match point and source point are on a vertical axis.
    if (a_p(1)-a_p(2))==0
        % Area of rectangle
        p = 0;
        A = (2*pi/seg_num)*(abs(a_p(2)-a_p(1))/seg_num);
        for z = a_p(1):(a_p(2)-a_p(1))/seg_num:a_p(2);
            p = p + 1;
            Zq(p,:) = a_p.*((eps_r-eps_l)*a_p(1)*(rho_m(s)*sin(phi_m)-...  
                           (eps_r-eps_l)*a_p(1)*cos(phi_m).*cos(phi_p)).^2 ...  
                           +(eps_r-eps_l)*a_p(1)*cos(phi_m).*sin(phi_p)).^2 ...  
                           +(eps_r-eps_l)*a_p(1)*sin(phi_m).*cos(phi_p)).^2 ...  
                           +(eps_r-eps_l)*a_p(1)*sin(phi_m).*sin(phi_p)).^2);  
        end
    
    %Then the match point and source point are on a DIFFERENT axis.
    else
        % Area of rectangle
        p = 0;
        A = (2*pi/seg_num)*(abs(a_p(2)-a_p(1))/seg_num);
        for z = a_p(1):(a_p(2)-a_p(1))/seg_num:a_p(2);
            p = p + 1;
            Zq(p,:) = (eps_r-eps_l)*a_p.*((eps_r-eps_l)*a_p(1)*(rho_m(s)*sin(phi_m)-...  
                           (eps_r-eps_l)*a_p(1)*cos(phi_m).*cos(phi_p)).^2 ...  
                           +(eps_r-eps_l)*a_p(1)*cos(phi_m).*sin(phi_p)).^2 ...  
                           +(eps_r-eps_l)*a_p(1)*sin(phi_m).*cos(phi_p)).^2 ...  
                           +(eps_r-eps_l)*a_p(1)*sin(phi_m).*sin(phi_p)).^2);
        end
    end
    Z_int = sum(sum(Zq))*A;
else
    flag = 0;
end
```

This is the number the integration w.r.t. phi_p is broken up into.
It's number should be left at approx. 100 for a short computation time.
It's value of phi for the source points.
It's defines Constants
It's value of phi at the match points.
if match point is on vertical surface.
else
for z=z_p(1):(z_p(2)-z_p(1))/seg_num:z_p(2);
   p=p+1;
   Zq(p,:)=-(eps_r-eps_l)*a_p(1)*abs(z_m(s)-z)./(4*pi*e0)./((sqrt((rho_m(s)*cos(phi_m)-a_p(1).*cos(phi_p)).^2+(rho_m(s).*sin(phi_m)-a_p(1).*sin(phi_p)).^2+(z-z_m(s)).^2)).^3);
end
end
%Then the source points are on a horizontal axis.
else
A=(2*pi/seg_num)*(abs(a_p(2)-a_p(1))/seg_num); %area of rectangle
if match point is on horizontal surface.
   if normal=1;
      for a_p=a_p(1):(a_p(2)-a_p(1))/seg_num:a_p(2);
         p=p+1;Zq(p,:)=-((eps_r-eps_l)*a_p*(rho_m(s)*sin(phi_m)-...a_p*sin(phi_p))./(4*pi*e0))./((sqrt((rho_m(s)*cos(phi_m)-...a_p.*cos(phi_p)).^2+(rho_m(s).*sin(phi_m)-a_p*...sin(phi_p)).^2+(z_p(1)-z_m(s)).^2)).^3);
      end
   else
      for a_p=a_p(1):(a_p(2)-a_p(1))/seg_num:a_p(2);
         p=p+1;Zq(p,:)=-(eps_r-eps_l)*a_p*abs(z_m(s)-z_p(1))./(4*pi*e0)./((sqrt((rho_m(s)*cos(phi_m)-...a_p.*cos(phi_p)).^2+(rho_m(s).*sin(phi_m)-a_p*...sin(phi_p)).^2+(z_p(1)-z_m(s)).^2)).^3);
      end
   end
end
end
Z_int=sum(sum(Zq))*A;
elseif flag==3;
%*********************************************************% This 'if' statements calculates the current contributions from
%the full wave region on the quasi-static regions on the
%conductor/dielectric interfaces.
%*********************************************************
   for m_current=1:1:3
      m=m_current;s=0;
      for n=8:1:(7+N1)
         s=s+1;n_current=n;Z(m_current,n_current)=((1/(4*pi*j*w*e0))*quad(inline(...
            ['(exp((-j*2*pi/15).*sqrt((',num2str(rho_m(m)),').^2+(z-',...num2str(z_m(m)),').^2))')./(sqrt((',num2str(rho_m(m)),').^2+(z-',...num2str(z_m(m)),').^2))']),z_full(s),z_full(s+1),tolerance));
      end
   end
else
   do nothing
end
APPENDIX E. SPHERICAL COORDINATE SYSTEM

The spherical coordinate system defined throughout the thesis can be seen in Figure 38. The point $P$ is described by the vector

$$\vec{P} = P_R \hat{a}_R + P_\theta \hat{a}_\theta + P_\phi \hat{a}_\phi$$

where $\hat{a}_R$, $\hat{a}_\theta$, and $\hat{a}_\phi$ are unit vectors in the $R$, $\theta$, and $\phi$ directions respectively. Line integration $d\vec{L}$ is described by

$$d\vec{L} = dR \hat{a}_R + Rd\theta \hat{a}_\theta + R\sin\theta d\phi \hat{a}_\phi$$

and surface integration $d\vec{S}$ is described by

$$d\vec{S} = R^2 \sin\theta \ d\theta \ d\phi \ (\pm \hat{a}_R) = R \sin\theta \ dR \ d\phi \ (\pm \hat{a}_\theta) = R \ dR \ d\theta \ (\pm \hat{a}_\phi).$$

Figure 38. Spherical Coordinate System.
APPENDIX F. ANSOFT SETUP

Figure 39 illustrates the wave port definition used in Ansoft to excite the monopole. The $x - y$ plane is located on the infinite ground plane. The radius of the wire is defined to be $0.0075 \, cm$ and the wave port opening is defined to be $0.008 \, cm$. The line of excitation is defined from the wave port opening at $0.008 \, cm$ to the edge of the wire at $0.0075 \, cm$ ($\Delta = -0.0005 \, cm$).

Convergence in Ansoft was achieved for a smooth surface monopole with a radius of $a=0.0075 \, cm$ and a length of 10 $cm$. The convergence time for the entire sweep was $\approx 32$hrs. Once an inductor with a smooth surface was added to this problem the computer was incapable of producing valid results due to hardware limitations. At this point it was decided to define a tetrahedron with 14 sides for each surface. Figure 40 shows the mesh Ansoft defined on the surface of the inductor. The tetrahedron definition of all surfaces in the problem significantly reduced the computation time.

![Figure 39. Ansoft Wave Port Definition.](image_url)
The space around the monopole was defined as a cube of air with dimensions 10 cm x 10 cm x 11 cm and centered at (0,0,5.5) cm. The input impedance of the monopole did not converge for regions of air smaller than a cube of this dimension. This problem definition provided the Ansoft results displayed in Figures 29-31.
APPENDIX G. QUICNEC

The flow chart for QUICNEC is shown in Figure 41. Several screen shots of the QUICNEC GUI can be seen in Figures 42-47. The code for the subroutines used in QUICNEC can be seen in [27].

Figure 41. QUICNEC Flow Chart.
Figure 42. QUICNEC GUI Introduction.

Figure 43. QUICNEC GUI Main Input Window.
Figure 44. QUICNEC GUI Surface Input Window.

Figure 45. QUICNEC GUI Source Input Window.
Figure 46. QUICNEC GUI Output Window.

Figure 47. QUICNEC GUI Help Window.