INTEGRALLY CLOSED ALMOST COMPLETE INTERSECTION IDEALS

CĂTĂLIN CIUPERCĂ

Abstract. Let $(A, \mathfrak{m})$ be a local ring. We study the existence and structure of the $\mathfrak{m}$-primary integrally closed ideals generated by $\dim A + 1$ elements.

Introduction

Let $(A, \mathfrak{m})$ be a local ring of dimension d and I an ideal of A. An element $x \in A$ is said to be integral over I if x satisfies an equation $x^n + a_1 x^{n-1} + \ldots + a_n = 0$ with $a_i \in I^i$. The set of all elements in A that are integral over I is an ideal $\overline{I}$, and the ideal I is called integrally closed if $I = \overline{I}$. In [3], Goto studied the $\mathfrak{m}$-primary integrally closed complete intersection ideals. It is shown that such ideals exist only when the ring is regular, and an $\mathfrak{m}$-primary complete intersection ideal in a regular local ring is integrally closed if and only if it contains $d-1$ regular parameters. Moreover, all the powers of such an ideal are integrally closed. The results are also extended to the case of complete intersection ideals of arbitrary dimension.

In this note we study the structure of the $\mathfrak{m}$-primary integrally closed ideals minimally generated by $d+1$ elements. First we observe that such ideals exist if and only if the maximal ideal of A is minimally generated by at most $d+1$ elements. When the ring A is regular, under the additional assumption that A contains a field, we prove that there exist regular parameters $a_1, \ldots, a_{d-2}$ contained in I such that IA' is integrally closed, where $A' = A/(a_1, \ldots, a_{d-2})$. The A'-ideal IA' is generated by three elements and the structure of integrally closed ideals generated by three elements in a two dimensional regular local ring is given by results of Noh [7]. We also prove the following theorem.

Theorem 0.1. *Let $(A, \mathfrak{m})$ be a regular local ring of dimension $d \geq 2$ and let I be an $\mathfrak{m}$-primary integrally closed ideal minimally generated by $d+1$ elements. Assume that A contains a field. Then the Rees algebra $\mathcal{R} = \oplus_{n \geq 0} I^n t^n$ is a Cohen-Macaulay normal domain and the associated graded ring $\mathcal{G} = \oplus_{n \geq 0} I^n / I^{n+1}$ is a Cohen-Macaulay ring with $a(\mathcal{G}) = 1 - d$.*

The a-invariant of G, denoted $a(\mathcal{G})$, is defined by $a(\mathcal{G}) = \sup\{i \mid \mathrm{H}_d^M(\mathcal{G})_i \neq 0\}$, where M is the maximal homogeneous ideal of $\mathcal{G}$. We refer the reader to [10, Chapter 5] for an exposition of the properties of this invariant in the context of Rees algebras and associated graded rings.

In the second case, when the embedding dimension of A is $d+1$, we prove the following.

Theorem 0.2. *Let $(A, \mathfrak{m})$ be a d-dimensional local ring with infinite residue field and maximal ideal minimally generated by $d+1$ elements, and let I be an $\mathfrak{m}$-primary almost complete intersection ideal. If I is integrally closed, then there exists a minimal set of generators $x, a_1, \ldots, a_d$ for $\mathfrak{m}$ such that either*

(1) *$I = (x^{i+1}, a_1, \ldots, a_{d-1}, a_d)$ with $x^i \notin \overline{(a_1, \ldots, a_d)}$, or*
(2) *$\mathfrak{m}^2 = \mathfrak{m}(x, a_1, \ldots, a_{d-1})$ and $I = (x^{i+1}, a_1, \ldots, a_{d-1}, z)$, for some element $z \in (x, a_1, \ldots, a_{d-1})$.*

Also, any ideal of type (1) is integrally closed. In particular, if the reduction number $r(\mathfrak{m})$ is at least 2 (for example, a hypersurface of multiplicity at least 3), then I is integrally closed if and only if I is of the type (1) described above.

In view of the constraint imposed on the class of ideals described in part (1), we also give an effective way of deciding whether $x_1^s \in \overline{(x_2, x_3, \ldots, x_{d+1})}$ in the hypersurface ring $R = k[x_1, x_2, \ldots, x_{d+1}]_{\mathfrak{m}}/(F)$ ($F \in (x_1, \ldots, x_{d+1})^2$).

As in [3], the $\mathfrak{m}$-full ideals are an important tool for the study of the integral closedness of $\mathfrak{m}$-primary almost complete intersection ideals.

Acknowledgement. The author thanks the referee for suggesting an improved version of Theorem 0.1 and for other comments that corrected a first version of this manuscript.

1. Preliminaries

Let $(A, \mathfrak{m})$ be a d-dimensional local ring with maximal ideal $\mathfrak{m}$. For an ideal I of A, $\mu(I)$ denotes the minimal number of generators of I and $\lambda(A/I)$ is the length of the A-module A/I. The embedding dimension of A, denoted $\operatorname{edim} A$, is the minimal number of generators of $\mathfrak{m}$. An $\mathfrak{m}$-primary ideal I is said to be almost complete intersection if $\mu(I) = d + 1$.

Originally introduced by Rees, the $\mathfrak{m}$-full ideals first appear in papers of Goto [3] and Watanabe [9].

Definition 1.1. Let $(A, \mathfrak{m})$ be a local ring and let I be an ideal of A. The ideal I is said to be $\mathfrak{m}$-full if there exists $x \in \mathfrak{m}$ such that $(I\mathfrak{m} : x) = I$.

Remark 1.2. Assume that the residue field $k = A/\mathfrak{m}$ is infinite. If I is an $\mathfrak{m}$-full ideal, then $I\mathfrak{m} : x = I$ for x general in $\mathfrak{m}$ (see [9, Appendix]). Also, it is proved in [3, Theorem 2.4] that if I is integrally closed, then either I is $\mathfrak{m}$-full or $I = \sqrt{(0)}$.

The following lemma plays an important role in many arguments in this note.

Lemma 1.3 (Watanabe [9]). *Let $(A, \mathfrak{m})$ be a local ring and I an $\mathfrak{m}$-primary ideal of A.*

(1) For every element x of $\mathfrak{m}$, we have

$$\mu(I) \leq \lambda(\mathfrak{m}I : x/\mathfrak{m}I) = \lambda(A/I + xA) + \mu(I + xA/xA).$$

(2) Suppose that I is $\mathfrak{m}$-full. Then $\mu(J) \leq \mu(I)$ for every ideal J containing I.

Using this result, Goto [3, Proposition 2.3] proved that an ideal I generated by a full system of parameters is $\mathfrak{m}$-full if and only if the ring is regular and $\lambda(I+\mathfrak{m}^2/\mathfrak{m}^2) \geq d-1$. Also, all the powers of an $\mathfrak{m}$-full parameter ideal are integrally closed ([3, Theorem 3.1]).

We start the study of the $\mathfrak{m}$-primary integrally closed almost complete intersection ideals with the following immediate observation.

Proposition 1.4. *Let $(A, \mathfrak{m})$ be a d-dimensional local ring and let I be an $\mathfrak{m}$-primary $\mathfrak{m}$-full ideal. If $\mu(I) = d+1$, then either R is regular or* $\operatorname{edim} A = d+1$.

Proof. By Lemma 1.3, we have $\mu(\mathfrak{m}) \leq \mu(I) = d+1$. □

2. Almost complete intersection ideals in regular local rings

In this section we study the structure of the $\mathfrak{m}$-primary integrally closed ideals with $\mu(I) = d+1$ in a regular local ring $(A, \mathfrak{m})$.

Proposition 2.1. *Let $(A, \mathfrak{m})$ be a d-dimensional regular local ring with infinite residue field, and let I be an $\mathfrak{m}$-primary ideal of A minimally generated by $d+1$ elements. Then I is $\mathfrak{m}$-full if and only if there exist regular parameters $x, y, a_1, \ldots, a_{d-2}$ such that $I + xA = (x, y^2, a_1, \ldots, a_{d-2})$.*

Proof. First we prove the direct implication. Let $x \in \mathfrak{m} \setminus \mathfrak{m}^2$ such that $(I\mathfrak{m} : x) = I$. By Lemma 1.3, we have

$$d+1 = \mu(I) = \lambda(A/I+xA) + \mu(I+xA/xA),$$

and since $\mu(I+xA/xA) \geq d-1$, this implies that $\lambda(A/I+xA) \leq 2$. If $\lambda(A/I+xA) = 1$, then $I+xA = \mathfrak{m}$, and hence $\mu(I+xA/xA) = d-1$. This contradicts the above equality, and therefore $\lambda(A/I+xA) = 2$. Choose $y \in \mathfrak{m}$ such that $\mathfrak{m} = I+(x,y)$ and $\mathfrak{m}y \subseteq I+xA$, so that $\mathfrak{m}^2 = \mathfrak{m}(I+(x,y)) \subseteq I+xA$. Considering the chain of ideals

$$\mathfrak{m}^2 \subset \mathfrak{m}^2 + xA \subset I + xA \subset I + (x,y) = \mathfrak{m},$$

we obtain $\lambda\big((I+xA)/(\mathfrak{m}^2+xA)\big) = d-2$, so there exists a saturated chain of ideals of length $d-2$

$$\mathfrak{m}^2 + xA \subset \mathfrak{m}^2 + (x, a_1) \subset \mathfrak{m}^2 + (x, a_1, a_2) \subset \ldots \subset \mathfrak{m}^2 + (x, a_1, \ldots, a_{d-2}) = I + xA.$$

Also, since $y \notin I+xA$ and $\mathfrak{m} = I+(x,y)$, the elements $x, y, a_1, \ldots, a_{d-2}$ form a regular system of parameters.

For the other implication, by Lemma 1.3, we have

$$d+1 = \mu(I) \leq \lambda(\mathfrak{m}I : x/\mathfrak{m}I) = \lambda(A/I+xA) + \mu(I+xA/xA) = 2 + d - 1 = d+1,$$

and hence $\mu(I) = \lambda(\mathfrak{m}I : x/\mathfrak{m}I)$. This implies that $I\mathfrak{m} : x = I$, so I is $\mathfrak{m}$-full. □

Remark 2.2. The ideals satisfying the equivalent conditions in the conclusion of Proposition 2.1 are not necessarily integrally closed. For example, let $R = k[[x,y]]$ and $I = (x^4, y^2, x^3y)$. Then $I + (x) = (x, y^2)$, but I is not integrally closed, as $x^2y \in \overline{I} \setminus I$.

Corollary 2.3. *Let $(A, \mathfrak{m})$ be a d-dimensional regular local ring and let I be an $\mathfrak{m}$-primary $\mathfrak{m}$-full ideal of A minimally generated by $d+1$ elements. Then $\dim_k(I + \mathfrak{m}^2/\mathfrak{m}^2) = d - 2$. Moreover, if I is integrally closed and the ring A contains a field, then there exist regular parameters $a_1, a_2, \ldots, a_{d-2} \in I$ such that IA' is an integrally closed ideal generated by three elements, where $A' = A/(a_1, \ldots, a_{d-2})$.*

Proof. Replacing A by $A[X]_{\mathfrak{m}[X]}$, we may assume that the residue field of A is infinite. By Proposition 2.1, there exist regular parameters $x, y, a_1, \ldots, a_{d-2}$ with $I + xA = (x, y^2, a_1, \ldots, a_{d-2})$. Note that we may also assume that $a_1, \ldots, a_{d-2} \in I$. This implies that $\mu(I + \mathfrak{m}^2/\mathfrak{m}^2) \geq d - 2$, and since I cannot be generated by d elements, we also have $\mu(I + \mathfrak{m}^2/\mathfrak{m}^2) \leq d - 2$.

Since every integrally closed ideal is $\mathfrak{m}$-full (Remark 1.2), the second part of the Corollary follows from the following lemma. □

Lemma 2.4. *Let $(A, \mathfrak{m})$ be a regular local ring containing a field, I an ideal of A, and $x \in \mathfrak{m} \setminus \mathfrak{m}^2$. Then*

$$\overline{\big((I + xA)/xA\big)} = \overline{(I + xA)}/xA.$$

Proof. By taking the completion of $(A, \mathfrak{m})$, we may assume that $A = k[[x_1, \ldots, x_d]]$. We may also assume that $x_1 = x$. Let $\pi : A \to A/xA$ be the canonical epimorphism and let $p : k[[x_2, \ldots, x_d]] = A/xA \to A$ be the k-algebra morphism defined by $p(x_i) = x_i$ $(i = 2, \ldots, d)$. Note that $\pi \circ p = id_{A/xA}$.

The inclusion "$\supseteq$"in the conclusion of the Lemma is clear. For the other inclusion, let $y \in S$ such that its image in A/xA is an element of $\overline{\big((I + xA)/xA\big)}$. Write an equation of integral independence and apply p. It follows that $y \in \overline{I + xA}$. □

When A is a regular local ring containing a field, Corollary 2.3 reduces the problem of describing the structure of $\mathfrak{m}$-primary almost complete intersection integrally closed ideals to the case when $\dim A = 2$. Then $\mu(I) = 3$, and [9, Theorem 4] implies that $\operatorname{ord}(I) = 2$, i.e., $I \subseteq \mathfrak{m}^2$ and $I \not\subseteq \mathfrak{m}^3$. By Zariski's theory of factorization of integrally closed ideals, either I is a product of two integrally closed parameter ideals or I is simple. In the first case, each of the two ideals is, up to a change of coordinates, of the form (x, y^n), where x, y is a regular system of parameters. In the second case, the structure of the simple integrally closed ideals of order 2 is given by Noh in [7].

Lemma 2.5. *Let $(A, \mathfrak{m})$ be a d-dimensional regular local ring containing a field, and let I be an ideal of A with* $\operatorname{edim}(A/I) \leq 2$. *Then $\overline{I^n} = I^{n-1}\overline{I}$ for all $n \geq 1$.*

Proof. Note that the assumption $\operatorname{edim}(A/I) \leq 2$ is equivalent to $\dim_k(I + \mathfrak{m}^2/\mathfrak{m}^2) \geq d - 2$.

We use induction on d. If $d = 1$, the statement is clear, and if $d = 2$, this is a result of Lipman and Teissier [6, Corollary 5.4].

We now consider the case when $d \geq 3$. We may assume that A has infinite residue field (if need be, replace A by $A[X]_{\mathfrak{m}[X]}$). Let $x \in I \setminus \mathfrak{m}^2$ be a sufficiently general element.

Let $y \in \overline{I^n}$ and denote by y' its image in $A/(x)$. Then $y' \in \overline{(I^n + (x))/(x))}$ and, by the induction hypothesis and Lemma 2.4, we have

$$y' \in \left(\frac{I^{n-1} + (x)}{(x)}\right)\overline{(I + (x))/(x))} = \left(\frac{I^{n-1} + (x)}{(x)}\right)(\overline{I}/(x)),$$

which implies that $y \in I^{n-1}\overline{I} + (x)$. Write $y = a + rx$ with $a \in I^{n-1}\overline{I}$ and $r \in A$. Then $rx \in \overline{I^n}$ and, since x is sufficiently general, we have $r \in (\overline{I^n} : x) = \overline{I^{n-1}}$, and hence $y \in I^{n-1}\overline{I} + x\overline{I^{n-1}} \subseteq I\overline{I^{n-1}}$. □

Remark 2.6. If $(A, \mathfrak{m})$ is a regular local ring containing a field, the above lemma can also be used to show that an $\mathfrak{m}$-primary complete intersection ideal is normal. Using different methods, this is proved by Goto [3, Theorem 3.1] without assuming that the ring contains a field.

Theorem 2.7. *Let $(A, \mathfrak{m})$ be a regular local ring of dimension $d \geq 2$ and let I be an $\mathfrak{m}$-primary integrally closed ideal minimally generated by $d + 1$ elements. Assume that A contains a field. Then the Rees algebra $\mathcal{R} = \oplus_{n \geq 0} I^n t^n$ is a Cohen-Macaulay normal domain and the associated graded ring $\mathcal{G} = \mathcal{G}_I(R) = \oplus_{n \geq 0} I^n/I^{n+1}$ is a Cohen-Macaulay ring with $a(\mathcal{G}) = 1 - d$.*

Proof. The normality of the Rees algebra follows from Lemma 2.5 and Corollary 2.3. Also, by [10, Theorem 5.1.23], it is enough to prove that $\mathcal{G}$ is Cohen-Macaulay and $a(\mathcal{G}) = 1 - d$.

We prove the statements by induction on the dimension of the ring. If $d = 2$, by [5, Theorem 3.2], it follows that $\mathcal{R}$ is Cohen Macaulay. In particular, $\mathcal{G}$ is also Cohen-Macaulay. Also, by [6, Proposition 5.5], I has reduction number one, and hence $a(\mathcal{G}) = -1$ (see [10, 5.1.26]).

Assume that $d \geq 3$ and let $x \in I \setminus \mathfrak{m}^2$ be a sufficiently general element. Since I is normal, we have $I^n : x = I^{n-1}$ for all n, so the image x^* of x in I/I^2 is a non-zero divisor on $\mathcal{G}$. Also, $\mathcal{G}/x^*\mathcal{G} \cong \mathcal{G}_{IR'}(R')$, where $R' = R/(x)$. By Lemma 2.4, the ideal IR' is integrally closed, and hence, by the induction hypothesis, the ring $\mathcal{G}$ is Cohen-Macaulay. Since $a(\mathcal{G}) + 1 = a(\mathcal{G}/x^*\mathcal{G})$, we also obtain $a(\mathcal{G}) = 1 - d$. □

Remark 2.8. If the ideal I is not primary to the maximal ideal, it is not necessarily true that an integrally closed almost complete intersection is normal. Let $R = k[[x, y, z]]$ ($\operatorname{char} k = 0$) and let P be the kernel of the natural k-algebra morphism from R to $k[[t^7, t^9, t^{10}]]$. It can be checked with Macaulay 2 [4] that $a = x^7z + x^2y^5 - 3x^3y^2z^2 + yz^5 \notin P^2$ and $a \in (P^2\mathfrak{m} : \mathfrak{m})$, so P^2 is not integrally closed.

3. Almost complete intersection ideals in rings of embedding dimension $d + 1$

In this section we describe the integrally closed $\mathfrak{m}$-primary almost complete intersection ideals when $\operatorname{edim} A = d + 1$. First we give a description of the larger class of $\mathfrak{m}$-full ideals.

Proposition 3.1. *Let $(A, \mathfrak{m})$ be a d-dimensional local ring with infinite residue field and* $\operatorname{edim} A = d+1$, *and let I be an $\mathfrak{m}$-primary ideal of A minimally generated by $d+1$ elements. Then I is $\mathfrak{m}$-full if and only if there exists a minimal set of generators $x, y, a_1, \dots, a_{d-1}$ of $\mathfrak{m}$ such that either*

(1) $I + xA = \mathfrak{m}$ or

(2) $\mathfrak{m}^2 = (x, a_1, \dots, a_{d-1})\mathfrak{m}$ and $I + xA = (x, a_1, \dots, a_{d-1})$.

Proof. First assume that I is $\mathfrak{m}$-full. Let $x \in \mathfrak{m} \setminus \mathfrak{m}^2$ such that $(I\mathfrak{m} : x) = I$. By Lemma 1.3, we have

$$d + 1 = \mu(I) = \lambda(A/I + xA) + \mu(I + xA/xA), \tag{3.1.1}$$

and since $\mu(I + xA/xA) \geq d - 1$, we obtain $\lambda(A/I + xA) \leq 2$. If $\lambda(A/I + xA) = 1$, then $I + xA = \mathfrak{m}$.

We now consider the case when $\lambda(A/I + xA) = 2$. Choose $y \in \mathfrak{m}$ such that $\mathfrak{m} = I + (x, y)$ and $\mathfrak{m}y \subseteq I + xA$. Then $\mathfrak{m}^2 = \mathfrak{m}(I + (x, y)) \subseteq I + xA$ and considering the chain of ideals

$$\mathfrak{m}^2 \subset \mathfrak{m}^2 + xA \subset I + xA \subset I + (x, y) = \mathfrak{m},$$

we get $\lambda(I + xA/\mathfrak{m}^2 + xA) = d - 1$, so there exists a saturated chain of ideals of length $d - 1$

$$\mathfrak{m}^2 + xA \subset \mathfrak{m}^2 + (x, a_1) \subset \mathfrak{m}^2 + (x, a_1, a_2) \subset \dots \subset \mathfrak{m}^2 + (x, a_1, \dots, a_{d-1}) = I + xA.$$

Since $y \notin I + xA$, it also follows that $x, y, a_1, \dots, a_{d-1}$ is a minimal set of generators of $\mathfrak{m}$.

So far we have proved that $I + xA = (x, y^2, a_1, \dots, a_{d-1})$. Since $\lambda(A/I + xA) = 2$, from (3.1.1) it follows that $\mu(I + xA/xA) = d - 1$. Consider the A/xA-ideal $K = (I + xA)/xA$. Let us observe that $a_1, \dots, a_{d-1}$ are linearly independent in $K/\mathfrak{m}K$. By contradiction, if they are linearly dependent, say $a_{d-1} \in \mathfrak{m}K + (a_1, \dots, a_{d-2})A/xA$, it follows that in A we have $a_{d-1} \in \mathfrak{m}(I + xA) + (a_1, \dots, a_{d-2}) + xA$, and hence $I + xA = (x, y^2, a_1, \dots, a_{d-2})$. Since $I + (x, y) = \mathfrak{m}$, we then have $\mathfrak{m} = (x, y, a_1, \dots, a_{d-2})$, contradicting $\mu(\mathfrak{m}) = d + 1$. So $a_1, \dots, a_{d-1}$ are linearly independent in $K/\mathfrak{m}K$. Since $\mu(K) = d - 1$ and $K = (y^2, a_1, \dots, a_{d-1})A/xA$, it follows that $y^2 \in \mathfrak{m}K + (a_1, \dots, a_{d-1})A/xA$, or equivalently, $y^2 \in \mathfrak{m}(I + xA) + (a_1, \dots, a_{d-1}) + xA$. This implies that $I + xA = (x, a_1, \dots, a_{d-1})$.

From the exact sequence

$$0 \to I/\mathfrak{m}I \to \mathfrak{m}/\mathfrak{m}I \xrightarrow{\cdot x} \mathfrak{m}/\mathfrak{m}I \to \mathfrak{m}/(\mathfrak{m}I + x\mathfrak{m}) \to 0,$$

we obtain $\mu(I) = \lambda(I/\mathfrak{m}I) = \lambda\left(\mathfrak{m}/(\mathfrak{m}I + x\mathfrak{m})\right)$. On the other hand, $\mu(I) = \mu(\mathfrak{m}) = \lambda(\mathfrak{m}/\mathfrak{m}^2) \leq \lambda\left(\mathfrak{m}/(\mathfrak{m}I + x\mathfrak{m})\right)$, and therefore $\mathfrak{m}^2 = \mathfrak{m}I + x\mathfrak{m}$.

We now prove that every ideal of type (1) or (2) is $\mathfrak{m}$-full.

By Lemma 1.3, we have

$$d + 1 = \mu(I) \leq \lambda(\mathfrak{m}I : x/\mathfrak{m}I) = \lambda(A/I + xA) + \mu(I + xA/xA).$$

If $I + xA = \mathfrak{m}$, then $\lambda(A/I + xA) = 1$ and $\mu(I + xA/xA) = d$, so $\lambda(\mathfrak{m}I : x/\mathfrak{m}I) = \mu(I) = \lambda(I/\mathfrak{m}I) = d + 1$. Then $I\mathfrak{m} : x = I$, so I is $\mathfrak{m}$-full. In the other case, when $I + xA = (x, a_1, \dots, a_{d-1})$ and $\mathfrak{m}^2 \subseteq I + xA$, we have $\lambda(A/I + xA) \leq 2$ and

$\mu(I + xA/xA) \leq d - 1$, so again $\lambda(\mathfrak{m}I : x/\mathfrak{m}I) = \mu(I) = d + 1$, and hence I is $\mathfrak{m}$-full. $\square$

Theorem 3.2. *Let $(A, \mathfrak{m})$ be a d-dimensional local ring with infinite residue field and* $\operatorname{edim} A = d + 1$, *and let I be an $\mathfrak{m}$-primary almost complete intersection ideal. If I is integrally closed, then there exists a minimal set of generators $x, a_1, \ldots, a_d$ for $\mathfrak{m}$ such that either*

1. *$I = (x^{i+1}, a_1, \ldots, a_{d-1}, a_d)$ with $x^i \notin \overline{(a_1, \ldots, a_d)}$, or*
2. *$\mathfrak{m}^2 = \mathfrak{m}(x, a_1, \ldots, a_{d-1})$ and $I = (x^{i+1}, a_1, \ldots, a_{d-1}, z)$, for some element $z \in (x, a_1, \ldots, a_{d-1})$.*

Also, any ideal of type (1) is integrally closed. In particular, if the reduction number $r(\mathfrak{m})$ is at least 2 (for example, a hypersurface of multiplicity at least 3), then I is integrally closed if and only if I is of the type (1) described above.

Proof. Assume that I is integrally closed. Then I is $\mathfrak{m}$-full, so there exists $x \in \mathfrak{m} \setminus \mathfrak{m}^2$ with $I\mathfrak{m} : x = I$ and I is of one of the forms described in Proposition 3.1.

First we consider the case when $I + xA = \mathfrak{m}$. Let i be such that $x^i \notin I$ but $x^{i+1} \in I$. Since $(I\mathfrak{m} : x) = I$, we have $x^{i+1} \notin I\mathfrak{m}$, so there exist $a_1, \ldots, a_d$ such that $I = (x^{i+1}, a_1, \ldots, a_d)$. Since $I + (x) = \mathfrak{m}$, the ideal $\mathfrak{m}$ is minimally generated by $x, a_1, \ldots, a_d$.

We now want to see when an ideal of the form $I_{i+1} := (x^{i+1}, a_1, \ldots, a_d)$ with $\mathfrak{m} = (x, a_1, \ldots, a_d)$ is integrally closed. First let us note that I_{i+1} is integrally closed if and only if $x^i \notin \overline{I_{i+1}}$. Indeed, if I_{i+1} is not integrally closed, then there exists $y \in \overline{I_{i+1}} \setminus I_{i+1}$. Choose a unit u in A such that $y = ux^t + b$ for some positive integer t and $b \in (a_1, \ldots, a_d)$. Then $x^t \in \overline{I_{i+1}} \setminus I_{i+1}$, hence $t \leq i$, and therefore $x^i \in \overline{I_{i+1}}$. The other implication is clear.

On the other hand, we have $x^i \in \overline{I_{i+1}}$ if and only if $x^i \in \overline{(a_1, \ldots, a_d)}$. Indeed, if $x^i \in \overline{I_{i+1}}$, then $\mathfrak{m}^{i+1} + (a_1, \ldots, a_d) = \mathfrak{m}(\mathfrak{m}^i + (a_1, \ldots, a_d)) + (a_1, \ldots, a_d)$ is a reduction of $\mathfrak{m}^i + (a_1, \ldots, a_d)$, hence $(a_1, \ldots, a_d)$ is a reduction of $\mathfrak{m}^i + (a_1, \ldots, a_d)$. This shows that $x^i \in \overline{(a_1, \ldots, a_d)}$.

In conclusion, if $k = \max\{s | x^s \notin \overline{(a_1, \ldots, a_d)}\}$, the ideal I_{i+1} is integrally closed for $i \in \{0, \ldots, k\}$ and not integrally closed for $i \geq k + 1$.

Next we consider the case when I is of the second type described in Proposition 3.1, i.e., there exist $x, y, a_1, \ldots, a_{d-1}$ generators for $\mathfrak{m}$ such that $\mathfrak{m}^2 = (x, a_1, \ldots, a_{d-1})\mathfrak{m}$ and $I + xA = (x, a_1, \ldots, a_{d-1})$. Let i be such that $x^{i+1} \in I$ and $x^i \notin I$; since $(I\mathfrak{m} : x) = I$, the element x^{i+1} is a minimal generator of I. Let $b_2, \ldots, b_{d+1}$ be such that $I = (x^{i+1}, b_2, \ldots, b_{d+1})$. Since $(x, b_2, \ldots, b_{d+1}) = (x, a_1, \ldots, a_{d-1})$ and $\mu(I) = d + 1$, one of the elements b_j is in the ideal generated by x and the other elements b_i. We may assume that $b_{d+1} \in (x, b_2, \ldots, b_d)$, and hence we have $\mathfrak{m} = (x, y, b_2, \ldots, b_d)$.

If the reduction number $r(\mathfrak{m}) \geq 2$, then I must be of the first type described in the statement and, as we have seen in the first part of the proof, any such ideal is integrally closed. As explained in the following remark, if A is Cohen-Macaulay, $r(\mathfrak{m}) = 1$ if and only if A is a hypersurface of multiplicity $\mathrm{e}(A) = 2$. $\square$

Remark 3.3. If $(A, \mathfrak{m})$ is a Cohen-Macaulay ring with infinite residue ring, it was proved by Abhyankar [1] that $\mathrm{e}(A) \geq \operatorname{edim} A - \dim A + 1$, where $\mathrm{e}(A)$ is the Hilbert-Samuel multiplicity of A . Moreover, Sally [8] proved that equality holds if and only if for some (hence every) minimal reduction K of $\mathfrak{m}$, $K\mathfrak{m} = \mathfrak{m}^2$. When A is a Cohen-Macaulay local ring of embedding dimension $d+1$, i.e., A is a hypersurface, this implies that ideals of type (2) described in Theorem 3.2 exist only when $\mathrm{e}(A) = 2$.

Note that ideals of type (1) exist in any local ring with $\operatorname{edim} A = d+1$ (the maximal ideal is such an example).

In the case when the reduction number of the maximal ideal is one, the characterization given by Theorem 3.2 is rather weak. As the following example shows, not every ideal of type (2) is integrally closed.

Example 3.4. Let $A = \mathbb{C}[[x,y]]/(xy + x^3 + y^3)$. Note that A is a one-dimensional Cohen-Macaulay domain, $\mathrm{e}(A) = 2$ and $\mathfrak{m}^2 = (x+y)\mathfrak{m}$. Then $\mu(\mathfrak{m}^n) = 2$ for all $n \geq 2$ and, by using the algorithm described in [2], it can be checked that $\mathfrak{m}$ is a normal ideal. On the other hand, in the ring $A = C[[x,y]]/(y^2)$, we have $\mathrm{e}(A) = 2$, $\mathfrak{m}^2 = x\mathfrak{m}$ and the ideal $K = \mathfrak{m}^2 = (x^2, xy)$ is of type (2), but is not integrally closed as $y \in \overline{K} \setminus K$.

For one-dimensional rings, a better description is given by the following Proposition.

Proposition 3.5. *Let $(A, \mathfrak{m})$ be a local Cohen-Macaulay ring of dimension one with $r(\mathfrak{m}) = 1$ and let I be an $\mathfrak{m}$-primary ideal with $\mu(I) = 2$. If I is integrally closed, then there exist generators x, y for $\mathfrak{m}$, a non-negative integer n, and an integrally closed ideal I' of type (1) such that $I = x^n I'$.*

Proof. As in the proof of Theorem 3.2, either I is of the type (1), in which case we take $n = 0$, or there exist x, y generators for $\mathfrak{m}$ and $z \in (x)$ such that $\mathfrak{m}^2 = x\mathfrak{m}$ and $I = (x^{i+1}, z)$. We may also assume that x is a not a zero-divisor. Let n be a positive integer such that $z = x^n z'$ and $z' \notin (x)$. Note that $i + 1 > n$, otherwise I would be principal. Then $I = x^n I'$ where $I' = (x^{i+1-n}, z')$, and since x is not a zero-divisor, I' is also integrally closed. Write $z' = \alpha x + \beta y^t$, with $\alpha, \beta \in A$ and β unit, so that $I' + (x) = (x, y^t)$. Then we must have $t = 1$, otherwise $y^t \in \mathfrak{m}^2 \subseteq (x)$, contradicting $z' \notin (x)$. In conclusion, I' is an ideal of type (1). $\square$

For a hypersurface ring, the following proposition gives an effective way of deciding when an ideal of the form $(x^{i+1}, a_1, \ldots, a_d)$ is integrally closed.

Proposition 3.6. *Let $S = k[x_1, x_2, \ldots, x_{d+1}]$, $R = S/(F)$, where $F \in \mathfrak{m}^2$, $\mathfrak{m} = (x_1, \ldots, x_{d+1})$, and let $I_{i+1} := (x_1^{i+1}, x_2, \ldots, x_{d+1})R$. Consider the weighted grading given by $\deg x_1 = 1$ and $\deg x_2 = \ldots = \deg x_{d+1} = i$ and write $F = F_t + F_{t+1} + \ldots + F_n$, where F_j is a homogeneous polynomial (in the above grading) of degree j ($F_t \neq 0$). Then I_{i+1} is integrally closed if and only if $F_t(x_1, 0, \ldots, 0) = 0$.*

Proof. Note that I_{i+1} is integrally closed if and only if $I_{i+1}R_{\mathfrak{m}}$ is integrally closed. Also, it follows from the proof of Theorem 3.2 that I_{i+1} is integrally closed if and only if $x_1^i \notin \overline{(x_2, \ldots, x_{d+1})R}$.

First assume that $x_1^i \in \overline{(x_2, \ldots, x_{d+1})R}$. Then

$$x_1^{in} + f_1 x_1^{i(n-1)} + \ldots + f_n = FH,$$

for some $f_j \in (x_2, \ldots, x_{d+1})^j$ and $H \in k[x_1, x_2, \ldots, x_{d+1}]$.

Using the homogeneous lexicographical monomial order with $x_1 < x_2 < \ldots < x_{d+1}$ and $\deg x_1 = 1$, $\deg x_2 = \ldots = \deg x_{d+1} = i$, the smallest monomial on the left hand side of the above equality is x_1^{in}. This implies that the homogeneous part of smallest degree of F must contain a term of the form $a_0 x_1^t$ with $a_0 \in k, a \neq 0$, i.e., $F_t(x_1, 0, \ldots, 0) \neq 0$.

Conversely, assume that $F_t(x_1, 0, \ldots, 0) \neq 0$, i.e., one of the terms of F_t is of the form $a_0 x_1^t$ with $a_0 \in k$, $a_0 \neq 0$. Denote $K = (x_2, \ldots, x_{d+1})$. For each $j \in \{t, \ldots, n\}$, we can write $F_j = a_{j-t} x_1^j + F_j'$ with $a_{j-t} \in k$ and $F_j' \in x_1^{t-i}K + x_1^{t-2i}K^2 + \cdots + x_1^{t-si}K^s + K^{s+1}$, where s is the integer part of t/i.

Since $F(x_1, \ldots, x_{d+1}) = 0$, we get an equation of the form

$$x_1^t(a_0 + a_1 x_1 + \cdots + a_{n-t} x_1^{n-t}) + g_1 x_1^{t-i} + \cdots + g_s x_1^{t-si} + g_{s+1} = 0,$$

where $g_j \in K^j$, $a_j \in k$, and s is the integer part of t/i. Note that $u = (a_0 + a_1 x_1 + \cdots + a_{n-t} x_1^{n-t})$ is a unit in $R_{\mathfrak{m}}$, so the equation

$$\left[x_1^t u + g_1 x_1^{t-i} + \cdots + g_s x_1^{t-si} + g_{s+1}\right]^i = 0$$

gives the integral dependence of x_1^i over $(x_2, \ldots, x_{d+1})$ in $R_{\mathfrak{m}}$. This shows that $x_1^i \in \overline{(x_2, \ldots, x_{d+1})R}$. $\square$

Remark 3.7. The ideals described in Theorem 3.2 are not necessarily normal. In fact, for a hypersurface ring, in [2], there are necessary and sufficient conditions for the maximal ideal $\mathfrak{m}$ to be normal.

References

[1] S. S. Abhyankar, *Local rings of high embedding dimension*, Amer. J. Math. **89** (1967), 1073–1077.

[2] P. Brumatti, A. Simis, W. V. Vasconcelos, *Normal Rees algebras*, J. Algebra **112** (1988), no. 1, 26–48.

[3] S. Goto, *Integral closedness of complete-intersection ideals*, J. Algebra **108** (1987), no. 1, 151–160.

[4] D. Grayson and M. Stillman, *Macaulay 2, a software system for research in algebraic geometry*, available at `http://www.math.uiuc.edu/Macaulay2`.

[5] C. Huneke and J. D. Sally, *Birational extensions in dimension two and integrally closed ideals*, J. Algebra **115** (1988), 481–500.

[6] J. Lipman and B. Teissier, *Pseudorational local rings and a theorem of Briançon-Skoda about integral closures of ideals*, Michigan Math. J. **28** (1981), no. 1, 97–116.

[7] S. Noh, *Valuation ideals of order two in 2-dimensional regular local rings*, Math. Nachr, **261/262** (2003), 123–140.

[8] J. D. Sally, *On the associated graded ring of a local Cohen-Macaulay ring*, J. Math. Kyoto Univ. **17** (1977), no. 1, 19–21.

[9] J. Watanabe, $\mathfrak{m}$-*full ideals*, Nagoya Math. J. **106** (1987), 101–111.

[10] W. V. Vasconcelos, Arithmetic of Blowup Algebras, Cambridge University Press, 1994.

Department of Mathematics, 300 Minard Hall, North Dakota State University, Fargo, ND 58105-5075

E-mail address: catalin.ciuperca@ndsu.edu