DEGREES OF MULTIPLICITY FUNCTIONS FOR EQUIMULTIPLE IDEALS

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Abstract. Let J be an equimultiple ideal of height a in a formally equidimensional local ring $(R, \mathfrak{m})$. If I is a proper ideal that contains J, we show that the degree of the multiplicity function $f_{J,I}(n) = \mathrm{e}(I^n/J^n)$ is at most a with equality if and only if J is not a reduction of I. As a consequence, we are able to define a unique filtration $J \subseteq J_{[a]} \subseteq \ldots \subseteq J_{[1]} \subseteq \overline{J}$ between the ideal J and its integral closure $\overline{J}$ with $J_{[k]}$ being the largest ideal containing J such that $\deg f_{J,I}(n) \leq a - k - 1$. Further results consider the ideal $J_{[1]}$ and its relation to the S_2-ification of the Rees algebra $R[Jt]$.

1. Introduction

Let $(R, \mathfrak{m})$ be a local noetherian ring and J an $\mathfrak{m}$-primary ideal of R. For n sufficiently large, the length $\lambda(R/J^n)$ becomes a polynomial function

$$P_J(n) = \mathrm{e}_0(J)\binom{n+d-1}{d} - \mathrm{e}_1(J)\binom{n+d-2}{d-1} + \cdots + (-1)^d\, \mathrm{e}_d(J)$$

of degree $d = \dim R$ whose normalized top degree coefficient $e_0(J)$ is the so-called Hilbert-Samuel multiplicity of J. The fundamental importance of the multiplicity $e_0(J)$ in the study of integral closures of ideals was firmly established by Rees [11] who showed that if $(R, \mathfrak{m})$ is formally equidimensional and $J \subseteq I$ are $\mathfrak{m}$-primary ideals, then J and I have the same integral closure (or equivalently, $J \subseteq I$ is a reduction) if and only if I and J have the same Hilbert-Samuel multiplicity. In view of this result, the integral closure $\overline{J}$ of an ideal J is the largest ideal containing J with the same multiplicity. Based on this interpretation, Shah [13] proved that for each $k \in \{1, \ldots, \dim R\}$ there exists a unique ideal J_k containing J maximal with the property of having the same first $(k+1)$ Hilbert coefficients $e_0, \ldots, e_k$.

If $J \subseteq I$ are not necessarily $\mathfrak{m}$-primary but $\lambda(I/J)$ is finite, Amao and Rees [1, 12] showed that the length $\lambda(I^n/J^n)$ is eventually a polynomial function $P_{J,I}(n)$ of degree at most $\dim R$.

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Moreover, $J \subseteq I$ is a reduction if and only if the degree of $P_{J,I}(n)$ is at most $\dim R-1$. Based on this result, Herzog, Puthenpurakal and Verma showed in [5] that Shah's original argument can be used to prove that there exists a largest ideal J_k containing J with $\lambda(J_k/J) < \infty$ such that the degree of $P_{J,J_k}(n)$ is at most $\ell(J) - k - 1$, where $\ell(J)$ is the analytic spread of J. However, it is worth noting that the finiteness constraint on the length makes the construction non-trivial (i.e. $J_k \neq J$) only in the case when $(J : \mathfrak{m}^\infty) \neq J$, or equivalently, $\mathfrak{m} \in \operatorname{Ass}(R/J)$.

In the case when $J \subseteq I$ are arbitrary, in [4] we considered the asymptotic behavior of the multiplicity $\mathrm{e}(I^n/J^n)$ of the R-module I^n/J^n. For n large enough, $\mathrm{e}(I^n/J^n)$ is a polynomial function $P_{J,I}(n)$ of degree at most $\dim R-t$, where t is the stabilizing value of the dimension $\dim R/(J^n : I^n)$. Moreover, if $J \subseteq I$ is a reduction, then the degree of $P_{J,I}(n)$ is at most $\dim R - t - 1$. In some particular cases (e.g. J integrally closed on the punctured spectrum [4, 2.6]), we were also able to show that the converse holds, i.e., the maximal degree $\dim R-t$ of the polynomial $P_{J,I}(n)$ is attained if and only if $J \subseteq I$ is not a reduction.

In this paper we start by considering the asymptotic behavior of $\mathrm{e}(I^n/J^n)$ in the case when J is an equimultiple ideal of height $a = \operatorname{ht} J = \ell(J)$. In this situation, we prove in Theorem 2.6 that the degree of $P_{J,I}(n)$ is at most $\ell(J)$, with equality if and only if $J \subseteq I$ is not a reduction. This result is essential for showing later in Theorem 3.1 that for an equimultiple ideal J, for each $k \in \{0, \ldots, a\}$ there exists a largest ideal $J_{[k]}$ containing J such that the degree of $P_{J,J_{[k]}}$ is at most $a-k-1$, extending Shah's construction to the case of equimultiple ideals.

In the last part of the paper we concentrate on the properties of the ideal $J_{[1]}$ and its relation with the homogeneous component of degree one of the S_2-ification of the Rees algebra $R[Jt]$ (Theorem 3.12). Recall that the S_2 property of Serre is one of the two conditions required for the normality of the Rees algebra $R[Jt]$, the other one being the R_1 condition (regularity in codimension one). Detecting the S_2 property is also important when one is looking at the Cohen-Macaulayness of the Rees algebra. Whenever the S_2 property of the Rees algebra is missing, under some mild conditions on the ring R, one is able to realize it in a finite birational extension of R, the so-called S_2-ification (or S_2-closure) of $R[Jt]$. The process of obtaining it, including some computational approaches, has been studied for example in [8]. If the ideal J is $\mathfrak{m}$-primary, in [2] the author showed that the homogeneous component of degree one of the S_2-ification of $R[Jt]$ is the largest ideal J_1 containing J such that the degree of the length function $\lambda(J_1^n/J^n)$ is at most $\dim R - 2$, which is exactly the first coefficient ideal of J defined by Shah. In the more general case of equimultiple ideals J, in this paper we are able to show that if $J \subseteq I$, $J \neq I$, and $R[Jt]$ satisfies Serre's property S_2, then the degree of the multiplicity function $\mathrm{e}(I^n/J^n)$ is either $\ell(J) - 1$ (when $J \subseteq I$ is a reduction) or $\ell(J)$ (Corollary 3.13). We conclude with several results regarding the homogeneous component of degree one of the S_2-ification of the Rees algebra $R[Jt]$ of an arbitrary ideal J.

2. Degrees of multiplicity functions

We begin by establishing some of the notations and conventions of this paper. If $(R, \mathfrak{m})$ is a local noetherian ring with maximal ideal $\mathfrak{m}$ and M is a finitely generated R-module of dimension $d = \dim M$, the multiplicity $\mathrm{e}(M) = \mathrm{e}_d(M)$ is the normalized leading coefficient of the Hilbert function

$$\lambda(M/\mathfrak{m}^n M) = \frac{\mathrm{e}(M)}{d!} n^d + \text{lower degree terms} \quad (n \gg 0).$$

For $t > \dim M$, we also set $\mathrm{e}_t(M) = 0$. We caution that in the literature (e.g. [7]) a different definition is sometimes used by setting $\mathrm{e}(M) = \mathrm{e}_{\dim R}(M)$, in which case the multiplicity of M is non-zero if and only if M and R have the same dimension. The possible conflict is easily avoided by considering M as a module over $R/\operatorname{Ann} M$.

If J is an ideal in the local ring $(R, \mathfrak{m})$, the analytic spread of J is defined by $\ell(J) = \dim R[Jt]/\mathfrak{m}R[Jt]$. If the residue field $R/\mathfrak{m}$ is infinite, $\ell(J)$ gives the minimal number of generators of every minimal reduction of J. It is also known that $\operatorname{ht} J \leq \operatorname{alt} J \leq \ell(J) \leq \dim R$, where $\operatorname{alt} J = \max\{\operatorname{ht} \mathfrak{p} \mid \mathfrak{p} \text{ minimal prime over } J\}$ and $\operatorname{ht} J$ is the height of the ideal J. The ideals that satisfy the condition $\ell(J) = \operatorname{ht} J$ are called equimultiple ideals.

Throughout this paper, if $h(n)$ is a numerical function that is eventually a polynomial function $P(n)$, we write $\deg f(n)$ to refer to the degree of $P(n)$. For all the other unexplained or undefined terminology we refer the reader to [7].

2.1. The asymptotic behavior of $\lambda(I^n/J^n)$. If $J \subseteq I$ are ideals in the local ring R and $\lambda(I/J) < \infty$, Amao [1] showed that $\lambda(I^n/J^n)$ is eventually a polynomial function $P_{J,I}(n)$. Rees [12] later proved that the degree of this polynomial is at most $\dim R$ and if J is a reduction of I, then the degree is at most $\dim R - 1$. Moreover, if $(R, \mathfrak{m})$ is formally equidimensional (i.e. all the minimal prime ideals of the completion $\widehat{R}$ have the same dimension), Rees showed that J is a reduction of I if and only if the degree of $P_{J,I}(n)$ is at most $\dim R - 1$. Let us observe that if J is a reduction of I, then the degree of $P_{J,I}(n)$ is in fact at most $\ell(J) - 1$. For this, note that if J is a reduction of I, then $\mathcal{M} = \oplus_{n\geq 1}(I^n/J^n)$ is a finitely generated module over $\mathcal{R}(J) = \oplus_{n\geq 0} J^n$. Moreover, since $\lambda(I/J) < \infty$, there exists k such that $\mathfrak{m}^k\mathcal{M} = 0$, so $\mathcal{M}$ is a finitely generated $\mathcal{R}(J)/\mathfrak{m}^k\mathcal{R}(J)$-module. This implies that $\lambda(I^n/J^n)$ is eventually a polynomial function of degree at most $\dim \mathcal{R}(J)/\mathfrak{m}^k\mathcal{R}(J) - 1 = \ell(J) - 1$. In conclusion, if $(R, \mathfrak{m})$ is formally equidimensional, $J \subseteq I$ and $\lambda(I/J) < \infty$, then the degree of $P_{J,I}(n)$ is either at most $\ell(J) - 1$ (in the case when $J \subseteq I$ is a reduction) or exactly $\dim R$ (in the case when $J \subseteq I$ is not a reduction).

2.2. The asymptotic behavior of $\mathrm{e}(I^n/J^n)$. If $J \subseteq I$ are ideals in the local ring R but the length $\lambda(I/J)$ is not necessarily finite, in [4] we considered the asymptotic behavior of the numerical function given by the multiplicity of the R-module I^n/J^n in order to characterize whether or not J is a reduction of I. We first observed that for $n \gg 0$ we have $\sqrt{(J^n : I^n)} = \sqrt{(J^{n+1} : I^{n+1})}$ ([4, 2.2]) and if t denotes the stabilizing value of the dimension $\dim R/(J^n : I^n)$, then $\mathrm{e}(I^n/J^n)$ is eventually a polynomial function $P_{J,I}(n)$ of degree at most $\dim R - t$. If J is a reduction of I, we also noted that the degree of this polynomial function is at most $\dim R - t - 1$. In fact, similarly to what we noted above, if $J \subseteq I$ is a reduction we can improve the upper bound of the degree of this polynomial by showing that

$$\deg P_{J,I}(n) \leq \min\{\dim R - t - 1, \ell(J) - 1\}. \tag{2.2.1}$$

To see this, set $K := \sqrt{(J^n : I^n)}$ for $n \gg 0$. From the associativity formula, for $n \gg 0$ we have

$$\mathrm{e}(I^n/J^n) = \sum_{\mathfrak{p}\supseteq K, \dim R/\mathfrak{p}=t} \mathrm{e}(R/\mathfrak{p})\, \lambda(I^n_{\mathfrak{p}}/J^n_{\mathfrak{p}}).$$

By the observation made at the end of (2.1), each numerical function $\lambda(I^n_{\mathfrak{p}}/J^n_{\mathfrak{p}})$ from the right-hand side of the above equality has degree at most $\ell(J_{\mathfrak{p}}) - 1$, and hence the degree of $\mathrm{e}(I^n/J^n)$ is bounded above by $\ell(J)-1$. It is worth noting that this upper bound is independent of I, unlike $\dim R - t - 1$.

Remark 2.3. With the same set-up as in (2.2), assume in addition that J is equimultiple. Since $J^n \subseteq K$ for $n \gg 0$, we have $\ell(J) = \operatorname{ht} J \leq \operatorname{ht} K \leq \dim R - t$, and therefore the inequality $\deg \mathrm{e}(I^n/J^n) \leq \ell(J) - 1$ is the only relevant part of (2.2.1) for reductions $J \subseteq I$ with J equimultiple. On the other hand, if J is not a reduction of I, then, as we show below in Lemma 2.4, we have $\operatorname{ht} J = \operatorname{ht} K$. Moreover, under the additional assumption that R is formally equidimensional, in Theorem 2.6 we prove that if J is not a reduction of I and J is equimultiple, then the degree of $\mathrm{e}(I^n/J^n)$ is exactly $\dim R - t = \operatorname{ht} K = \operatorname{ht} J$.

Lemma 2.4. *Let $(R, \mathfrak{m})$ be a local ring and $J \subseteq I$ proper ideals of R with J equimultiple. Assume that J is not a reduction of I. Then*

$$\operatorname{ht} J = \operatorname{ht}(\overline{J} : I) = \operatorname{ht}(J^n : I^n) \text{ for all } n \geq 1.$$

Proof. Let $\mathfrak{p}$ be a minimal prime over $(\overline{J} : I)$. If $I = (x_1, \ldots, x_k)$, then $\mathfrak{p}$ is minimal over $(\overline{J} : x_i)$ for some i, and hence $\mathfrak{p} \in \operatorname{Ass}(R/\overline{J})$. Since J is equimultiple, a result of Ratliff [9, Theorem 2.12] shows that every prime in $\operatorname{Ass}(R/\overline{J})$ is minimal, so $\mathfrak{p}$ is minimal over J. On the other hand, we also have $\operatorname{ht} J = \ell(J) \geq \operatorname{alt} J \geq \operatorname{ht} \mathfrak{p}$, so $\operatorname{ht} J = \operatorname{ht} \mathfrak{p}$ and therefore $\operatorname{ht} \mathfrak{p} = \operatorname{ht}(\overline{J} : I) = \operatorname{ht}(J : I) = \operatorname{ht} J$. Similarly, for any positive integer n, the ideal J^n is equimultiple and J^n is not a reduction of I^n, and hence, for every minimal prime ideal $\mathfrak{p}$ over $(\overline{J^n} : I^n)$, we have $\operatorname{ht} \mathfrak{p} = \operatorname{ht}(\overline{J^n} : I^n) = \operatorname{ht}(J^n : I^n) = \operatorname{ht} J$. $\square$

Remark 2.5. Note that if J is equimultiple with $\ell(J) < \dim R$ and $J \subseteq I$ with $\lambda(I/J) < \infty$, then $J \subseteq I$ is a reduction. Indeed, by the previous lemma, if J is not a reduction of I, then $\operatorname{ht} J = \operatorname{ht}(J : I) = \dim R$.

Theorem 2.6. *Let $(R, \mathfrak{m})$ be a formally equidimensional local ring and $J \subseteq I$ proper ideals of R with J equimultiple. Let $f(n) = \mathrm{e}(I^n/J^n)$. The following are true.*

(a) *If $J \subseteq I$ is a reduction, then* $\deg f(n) \leq \ell(J) - 1$.

(b) *If $J \subseteq I$ is not a reduction, then* $\deg f(n) = \ell(J)$.

Proof. Let t denote the stabilizing value of $\dim R/(J^n : I^n)$. If $J \subseteq I$ is a reduction, as we already explained in 2.2, the degree of $f(n)$ is at most $\ell(J) - 1$. Assume that $J \subseteq I$ is not a reduction. In this case we will show that the degree of $f(n)$ is exactly $\dim R - t = \operatorname{ht} J$, where the last equality follows from Lemma 2.4. Choose N such that $K := \sqrt{(J^n : I^n)} = \sqrt{(J^{n+1} : I^{n+1})}$ for $n \geq N$. From the associativity formula, for $n \geq N$ we have:

$$\mathrm{e}(I^n/J^n) = \sum_{\mathfrak{q} \supseteq K, \dim R/\mathfrak{q} = t} \mathrm{e}(R/\mathfrak{q})\, \lambda(I^n_{\mathfrak{q}}/J^n_{\mathfrak{q}}). \tag{2.6.1}$$

As in the proof of Lemma 2.4, let $\mathfrak{p}$ be a minimal prime over $(\overline{J^N} : I^N)$. Then $\dim R/\mathfrak{p} = \dim R - \operatorname{ht} \mathfrak{p} = \dim R - \operatorname{ht}(J^N : I^N) = t$, so $\mathfrak{p}$ is one of the prime ideals that contributes with the term $\mathrm{e}(R/\mathfrak{p})\, \lambda(I^n_{\mathfrak{p}}/J^n_{\mathfrak{p}})$ on the right-hand side of (2.6.1). On the other hand, since $\mathfrak{p} \supseteq (\overline{J^N} : I^N)$, the ideal $J_{\mathfrak{p}}$ cannot be a reduction of $I_{\mathfrak{p}}$, in which case, by the result of Rees [12, 2.1] which we explained in 2.1, the degree of the function $\lambda(I^n_{\mathfrak{p}}/J^n_{\mathfrak{p}})$ is exactly $\dim R - t$. Since all the other terms in (2.6.1) have degree at most $\dim R - t$, this shows that the degree of $\mathrm{e}(I^n/J^n)$ is exactly $\dim R - t$. $\square$

3. Coefficient ideals for equimultiple ideals

If J is an $\mathfrak{m}$-primary ideal in local ring $(R, \mathfrak{m})$ of dimension d, for $n \gg 0$ the length $\lambda(R/J^n)$ becomes a polynomial function

$$P_J(n) = \mathrm{e}_0(J)\binom{n+d-1}{d} - \mathrm{e}_1(J)\binom{n+d-2}{d-1} + \cdots + (-1)^d\, \mathrm{e}_d(J).$$

If R is formally equidimensional and $\dim R > 0$, Shah [13] proved that for each $k \in \{1, \ldots, d\}$ there exists a unique ideal J_k containing J maximal with the property that $\mathrm{e}_i(J_k) = \mathrm{e}_i(J)$ for $0 \leq i \leq k$, or equivalently, $\deg \lambda(J^n_k/J^n) \leq d - k - 1$. The ideal J_k was called the k-th coefficient ideal of J.

In this section we define the concept of coefficient ideals for an equimultiple ideal J of height a in a formally equidimensional local ring $(R, \mathfrak{m})$. Using the numerical information of the degree functions studied in Section 2, we construct a family of ideals $J \subseteq J_a \subseteq J_{a-1} \subseteq \ldots \subseteq J_1 \subseteq \overline{J}$ that generalizes the above mentioned construction of the coefficient ideals of an $\mathfrak{m}$-primary ideal.

Theorem 3.1. *Let $(R, \mathfrak{m})$ be a formally equidimensional local ring of positive dimension and let J be an equimultiple ideal of R of height a. For each $k \in \{0, \ldots, a\}$ let*

$$\mathcal{L}_k = \mathcal{L}_k(J) = \{L \mid L \text{ ideal of } R, L \supseteq J, \text{and } \deg \mathrm{e}(L^n/J^n) \leq a - k - 1\}.$$

Then, for each k, there exists a unique maximal element $J_{[k]}$ of $\mathcal{L}_k$. We call $J_{[k]}$ the k-th coefficient ideal of J.

Proof. The proof we present is based, in essence, on the same idea used by Shah [13] in the case of $\mathfrak{m}$-primary ideals. We begin by making the crucial observation that if $L \in \mathcal{L}_k$, then $\deg \mathrm{e}(L^n/I^n) \leq a-1$ and therefore, by Theorem 2.6, J is a reduction of L. In fact, Theorem 2.6 shows that $L \in \mathcal{L}_0$ if and only if $J \subseteq L$ is a reduction, so the unique maximal element of $\mathcal{L}_0$ is the integral closure $\overline{J}$ of J.

To prove that each $\mathcal{L}_k$ has a unique maximal element we will show that if $K, L \in \mathcal{L}_k$, then $K + L \in \mathcal{L}_k$. Since $J \subseteq K \subseteq K + L$ it follows that $K \subseteq K + L$ is a reduction, so there exists r such that $(K+L)^n = (K+L)^r K^{n-r}$ for every $n \geq r$. Let t be the stabilizing value of the dimension $\dim R/(J^n : (K+L)^n)$. Then, for $n \gg 0$ we have:

$$\begin{aligned}
\mathrm{e}((K+L)^n/J^n) &= \mathrm{e}_t((K+L)^n/J^n) = \mathrm{e}_t((L^r K^{n-r} + L^{r-1}K^{n-r+1} + \cdots + LK^{n-1} + K^n)/J^n) \\
&\leq \mathrm{e}_t\left(\bigoplus_{i=0}^{r} (L^i K^{n-i}/J^n)\right) \\
&= \sum_{i=0}^{r} \mathrm{e}_t(L^i K^{n-i}/J^n) \qquad \text{(some terms may be equal to 0)} \\
&= \sum_{i=0}^{r} [\mathrm{e}_t(L^i K^{n-i}/L^i J^{n-i}) + \mathrm{e}_t(L^i J^{n-i}/J^n)].
\end{aligned}$$

For each $i \in \{0, \ldots, r\}$ we have

$$\mathrm{e}_t(L^i J^{n-i}/J^n) \leq \mathrm{e}_t(L^n/J^n)$$

and

$$\mathrm{e}_t(L^i K^{n-i}/L^i J^{n-i}) \leq \mu(L^i)\, \mathrm{e}_t(K^{n-i}/J^{n-i}),$$

where $\mu(L^i)$ denotes the minimal number of generators of L^i. Note that t is an upper bound for the dimensions of all the quotient modules involved here. Since $\mathrm{e}_t(L^n/J^n) \leq \mathrm{e}(L^n/J^n)$,

$e_t(K^{n-i}/J^{n-i}) \leq e(K^{n-i}/J^{n-i})$ for each $i = 0, \ldots, r$, and both $e(L^n/J^n)$ and $e(K^{n-i}/J^{n-i})$ are polynomial functions of degree at most $a-k-1$ for $n \gg 0$, it follows that $e((K+L)^n/J^n)$ is bounded above by a polynomial function of degree at most $a-k-1$ for $n \gg 0$, and hence $K + L \in \mathcal{L}_k$, finishing the proof. $\square$

Remark 3.2. As already mentioned in the proof of the previous theorem, the ideal $J_{[0]}$ is simply the integral closure $\overline{J}$ of the ideal J.

Remark 3.3. In [5], a different construction for coefficient ideals of arbitrary ideals was introduced. The authors obtained a chain of ideals $J \subseteq J_a \subseteq J_{a-1} \subseteq \ldots \subseteq J_1 \subseteq \overline{J} \cap (J : \mathfrak{m}^\infty) = q(J)$ where J_k is the largest ideal containing J with $\lambda(J_k/J) < \infty$ and $\deg \lambda(J_k^n/J^n) \leq a - k - 1$. While this construction applies to an arbitrary ideal, only the case when $\mathfrak{m} \in \operatorname{Ass}(R/J)$ is of interest, for otherwise $q(J) = J$.

3.4. The S_2-ification of a noetherian domain. We recall a few properties of the S_2-ification of a noetherian domain. We refer the reader to [6, 2, 3] for more detailed accounts. If R is a noetherian domain, we say that S is an S_2-ification of R if the following properties are satisfied: (a) S is a finite birational extension of R; (b) S satisfies the S_2 property of Serre; and (c) S is minimal among the extensions of R satisfying (a) and (b). The S_2-ification of a noetherian domain does not always exist, but if it does, it is unique. More precisely, the noetherian domain R has an S_2-ification if and only if $\bigcap_{\operatorname{ht} p=1} R_\mathfrak{p}$ is a finite extension of R, in which case $\tilde{R} = \bigcap_{\operatorname{ht} p=1} R_\mathfrak{p}$ is the S_2-ification of R. The S_2-ification exists for a large class of noetherian domains. For example, if R is a formally equidimensional analytically unramified local domain, then R has an S_2-ification. Other examples of noetherian domains that have an S_2-ification include the local domains that have a canonical module ω, in which case $\operatorname{Hom}_R(\omega, \omega)$ is the S_2-ification of R.

In Theorem 3.1 we defined the first coefficient ideal of an equimultiple ideal J to be the largest ideal I containing J such that $\deg e(I^n/J^n) \leq \ell(J) - 2$. In the course of the study of the S_2-ification of the Rees algebra of an ideal J, in [3] we introduced the following construction associated with an ideal (see Definition 3.5), which we also called "first coefficient ideal". As we will see later, the two concepts are identical in the case of $\mathfrak{m}$-primary ideals.

In the more general case of equimultiple ideals, we will explore the connection between them in Theorem 3.12.

Definition 3.5 (First coefficient ideals as in [3])**.** Let A be a noetherian ring and J an ideal of A. Define

$$J_{\{1\}} = \bigcup (J^{n+1} : a),$$

where the union is taken over all $n \geq 1$ and all $a \in J^n \setminus J^{n+1}$ such that the image a^* of a in J^n/J^{n+1} is part of a system of parameters of the associated graded ring $\mathcal{G}(J) = \bigoplus_{n\geq 0} J^n/J^{n+1}$.

Remark 3.6. As proved in [3, Proposition 3.5], for each ideal J there exists a fixed integer m and a fixed element $a \in J^m \setminus J^{m+1}$ with a^* part of a system of parameters on $\mathcal{G}(J)$ such that

$$J_{\{1\}} = (J^{m+1} : a).$$

Remark 3.7. If J is an $\mathfrak{m}$-primary ideal in a local ring R, the ideal $\mathfrak{m}\mathcal{G}(J)$ is nilpotent. Therefore, in Definition 3.5, the union is taken over all n and all a such that $a^* \in J^n/J^{n+1}$ is part of a system of parameters on $\mathcal{G}(J)/\mathfrak{m}\mathcal{G}(J)$, or equivalently, a is extendable to a minimal reduction of J^n. By [13, Theorem 2], this means that in the case of an $\mathfrak{m}$-primary ideal J, the ideal $J_{\{1\}}$ coincides with the concept of first coefficient ideal defined by Shah, i.e. $J_{\{1\}}$ is the largest ideal I containing J such that the degree of $\lambda(I^n/J^n)$ is at most $\dim R - 2$.

3.8. The S_2-ification of a Rees algebra. The concept introduced in Definition 3.5 was motivated by the following result (see [3, Theorem 3.4] and the discussion preceding it). Let $(R, \mathfrak{m})$ be a formally equidimensional, analytically unramified local domain with infinite residue field and positive dimension and let J be an arbitrary ideal of R. Let $\widetilde{\mathcal{R}} = \bigoplus_{n\in\mathbb{Z}} J_n t^n \subseteq Q(R)[t, t^{-1}]$ be the S_2-ification of $R[Jt, t^{-1}]$. Then $J_n \cap R = J^n_{\{1\}}$ for every $n \geq 1$. We conclude by mentioning that if $\operatorname{ht} J \geq 2$, the S_2-ifications of the algebras $R[Jt, t^{-1}]$ and $R[Jt]$ have the same homogeneous components in positive degree [2, 2.6].

3.9. Properties of $J_{\{1\}}$. Let J be an ideal in a formally equidimensional local ring $(R, \mathfrak{m})$ We collect here several properties of $J_{\{1\}}$ that will be needed later. With the exception of (c), they all follow immediately from the interpretation of $J_{\{1\}}$ given by the S_2-ification of

Rees algebra, under the additional assumption that R is analytically unramified local domain with infinite residue field and positive dimension.

(a) If $J \subseteq I \subseteq J_{\{1\}}$, then $J \subseteq I$ is a reduction.
(b) $(J_{\{1\}})_{\mathfrak{p}} = (J_{\mathfrak{p}})_{\{1\}}$ for every prime ideal $\mathfrak{p}$.
(c) If $J \subseteq I$, then $I \subseteq J_{\{1\}}$ if and only if $\dim_{\mathcal{G}(J)} \bigoplus_{n\geq 0}(IJ^n/J^{n+1}) < \dim R$. ([3, Proposition 3.6])
(d) If $J \subseteq I$ is a reduction, then $J_{\{1\}} \subseteq I_{\{1\}}$.
(e) $(J_{\{1\}})_{\{1\}} = J_{\{1\}}$.
(f) If $J^* = \bigcup_{i\geq 1}(J^{i+1} : J^i)$ is the Ratliff-Rush closure of J, then $J^* \subseteq J_{\{1\}}$. (See, for example, the proof of [8, 2.10].)

A direct proof of (a) using only the definition of $J_{\{1\}}$ is given in [3, Proposition 3.8] for any ideal in a formally equidimensional local ring. Similarly, (b) can also be obtained directly from the definition.

We denote by J^{unm} the intersection of the primary components of J corresponding to its minimal primes. A slightly weaker version of the following result was proved by the author in [2, Proposition 2.10]. The original proof is modified by using the extended Rees algebra $R[Jt, t^{-1}]$.

Lemma 3.10. *Let $(R, \mathfrak{m})$ be a formally equidimensional, analytically unramified local domain with infinite residue field and of positive dimension, and let J be an equimultiple ideal of R. Then*

$$J^{unm} \subseteq J_{\{1\}}.$$

In particular,

$$\mathrm{Ass}(R/J_{\{1\}}) = \mathrm{Min}(R/J).$$

Proof. Let $\widetilde{\mathcal{R}} = \oplus_{n\in\mathbb{Z}} J_n t^n \subseteq Q(R)[t, t^{-1}]$ be the S_2-ification of the extended Rees algebra $\mathcal{R} = R[Jt, t^{-1}]$. As explained in 3.8, we have $J_n \cap R = (J^n)_{\{1\}}$ for all $n \geq 1$. Since $J^{\mathrm{unm}} = J$ when J is $\mathfrak{m}$-primary, we may assume that $\mathrm{ht}\, J \leq \dim R - 1$. Let $J = (\mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_k) \cap (\mathfrak{q}_{k+1} \cap \ldots \cap \mathfrak{q}_s)$ be an irredundant primary decomposition of J where $\mathfrak{p}_1 = \sqrt{\mathfrak{q}_1}, \ldots, \mathfrak{p}_k = \sqrt{\mathfrak{q}_k}$ are the minimal prime ideals of J, so that $J^{\mathrm{unm}} = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_k$. Suppose $J^{\mathrm{unm}} \not\subseteq J_{\{1\}}$ so that there exists $\mathfrak{p}$

prime ideal such that $(J^{\mathrm{unm}})_{\mathfrak{p}} \nsubseteq (J_{\{1\}})_{\mathfrak{p}}$. Choose $\mathfrak{p}$ minimal with this property. By localizing at $\mathfrak{p}$ we reduce to the case when $\operatorname{Supp}((J^{\mathrm{unm}}+J_{\{1\}})/J_{\{1\}}) = \{\mathfrak{m}\}$. Note that after localization the ideal J is still equimultiple and $\operatorname{ht} J < \dim R$. Indeed, since $(J^{\mathrm{unm}})_{\mathfrak{p}_i} = J_{\mathfrak{p}_i}$ for all $i \leq k$, the prime ideal $\mathfrak{p}$ is not minimal over J. Since $(J^{\mathrm{unm}} + J_{\{1\}})/J_{\{1\}}$ has finite length, we can choose r such that $\mathfrak{m}^r J^{\mathrm{unm}} \subseteq J_{\{1\}}$, so that for every $x \in J^{\mathrm{unm}}$ we have $xt \in Q(R)[t, t^{-1}]$ and $\mathfrak{m}^r xt \subseteq \widetilde{\mathcal{R}}$. Note that this also means that $(\mathfrak{m}^r, t^{-1})\widetilde{\mathcal{R}}xt \subseteq \widetilde{\mathcal{R}}$. On the other hand, we have

$$\dim R > \ell(J) = \dim \mathcal{R}/(\mathfrak{m}^r, t^{-1})\mathcal{R} \geq \dim \mathcal{R}/(\mathfrak{m}^r, t^{-1})\widetilde{\mathcal{R}} \cap \mathcal{R} = \dim \widetilde{\mathcal{R}}/(\mathfrak{m}^r, t^{-1})\widetilde{\mathcal{R}},$$

so $\operatorname{ht}(\mathfrak{m}^r, t^{-1})\widetilde{\mathcal{R}} \geq 2$. This implies that $xt \in \widetilde{(\widetilde{\mathcal{R}})} = \widetilde{\mathcal{R}}$, and therefore $x \in J_1 \cap R = J_{\{1\}}$. Finally, we also obtain $(J_{\{1\}})^{\mathrm{unm}} \subseteq (J_{\{1\}})_{\{1\}} = J_{\{1\}}$, and therefore $J_{\{1\}}$ is unmixed. $\square$

Lemma 3.11. *Let R be a noetherian ring, $J \subseteq I$ proper ideals of R, and let $r \in \mathbb{N}$. Then*

$$\sqrt{(J^n : I^n)} = \sqrt{(J^n : I^r J^{n-r})} \quad \text{for } n \gg 0.$$

Proof. For an ideal L, let $L^* = \bigcup_{i \geq 1}(L^{i+1} : L^i)$ denote the Ratliff-Rush closure of L [10]. As mentioned in 2.2, we can set $K := \sqrt{(J^n : I^n)}$ for $n \gg 0$. For $\alpha \in (J^n : I^r J^{n-r})$ we have $\alpha I^r \subseteq (J^n : J^{n-r}) \subseteq (J^r)^*$ ([10, 2.3.1]) and therefore $\alpha^n I^{rn} \subseteq ((J^r)^*)^n = J^{rn}$ for $n \gg 0$ [10, 2.1]. This implies that $\alpha \in K$, finishing the proof. $\square$

The next theorem relates the two concepts of first coefficient ideals in the case of equimultiple ideals.

Theorem 3.12. *Let $(R, \mathfrak{m})$ be a formally equidimensional, analytically unramified local domain with infinite residue field and of positive dimension, and let J be an equimultiple ideal of R. Then*

$$J_{[1]} \subseteq J_{\{1\}}.$$

Proof. For an ideal I, we will prove that $J \subseteq I \subseteq J_{[1]}$ implies $J \subseteq I \subseteq J_{\{1\}}$. Assume that $J \subseteq I \subseteq J_{[1]}$ but $I \nsubseteq J_{\{1\}}$. Then $J \subseteq I$ is a reduction. As before, set $K := \sqrt{(J^n : I^n)}$ for $n \gg 0$ and $t = \dim R/K$. Note that $K = \sqrt{(J^{n+1} : IJ^n)}$ for $n \gg 0$ by Lemma 3.11. Then, for $n \gg 0$, we have $\dim(I^n/J^n) = \dim(IJ^n/J^{n+1})$, hence $\mathrm{e}(I^{n+1}/J^{n+1}) \geq \mathrm{e}(IJ^n/J^{n+1})$, which implies that $\deg \mathrm{e}(IJ^n/J^{n+1}) \le \ell(J) - 2$. Since

$$\mathrm{e}(IJ^n/J^{n+1}) = \sum_{\mathfrak{p} \supseteq K, \dim R/\mathfrak{p} = t} \mathrm{e}(R/\mathfrak{p})\, \lambda(I_\mathfrak{p} J^n_\mathfrak{p}/J^{n+1}_\mathfrak{p}) \text{ for } n \gg 0, \tag{3.12.1}$$

we obtain that $\deg \lambda(I_\mathfrak{p} J^n_\mathfrak{p}/J^{n+1}_\mathfrak{p}) \le \operatorname{ht} J - 2$ for all the prime ideals $\mathfrak{p} \supseteq K$ with $\dim R/\mathfrak{p} = \dim R/K$.

Since $I \nsubseteq J_{\{1\}}$ we can take $\mathfrak{q}$ a minimal prime over $(J_{\{1\}} : I)$. Then $\mathfrak{q}$ is minimal over $(J_{\{1\}} : x)$ for some $x \in I$, hence $\mathfrak{q} \in \operatorname{Ass}(R/J_{\{1\}})$. Since $J_{\{1\}}$ has no embedded primes by Lemma 3.10, $\mathfrak{q}$ is a minimal prime over J. Then $\operatorname{ht} J = \ell(J) \ge \operatorname{alt} J \ge \operatorname{ht} \mathfrak{q}$, so $\operatorname{ht} \mathfrak{q} = \operatorname{ht} J$. From $J \subseteq (J^{n+1} : IJ^n) \subseteq (J^* : I) \subseteq (J_{\{1\}} : I)$ (see 3.9 (f)) we also get $\operatorname{ht} \mathfrak{q} = \operatorname{ht} K$. As R is formally equidimensional, $\mathfrak{q}$ is therefore one of the prime ideals that appear in the summation of (3.12.1), and hence $\deg \lambda(I_\mathfrak{q} J^n_\mathfrak{q}/J^{n+1}_\mathfrak{q}) \le \operatorname{ht} J - 2 = \operatorname{ht} \mathfrak{q} - 2$. On the other hand, since $I_\mathfrak{q} \nsubseteq (J_{\{1\}})_\mathfrak{q} = (J_\mathfrak{q})_{\{1\}}$ we have $\dim_{\mathcal{G}(J_\mathfrak{q})}(\bigoplus_{n \ge 0} I_\mathfrak{q} J^n_\mathfrak{q}/J^{n+1}_\mathfrak{q}) = \dim R_\mathfrak{q}$ (by 3.9 (c)), which implies that $\lambda(I_\mathfrak{q} J^n_\mathfrak{q}/J^{n+1}_\mathfrak{q})$ is eventually a polynomial function of degree $\dim R_\mathfrak{q} - 1$, reaching a contradiction.

We conclude the proof by noting that the unmixedness of $J_{\{1\}}$ was essential in the proof. The lack of a similar property for $J_{[1]}$ prevents one from obtaining the reverse containment (and thus equality) in the statement of the theorem. □

From Theorem 2.6 we know that if $J \subseteq I$, $J \ne I$ and J is equimultiple and integrally closed, then $\deg \mathrm{e}(I^n/J^n) = \ell(J)$. If J is not necessarily integrally closed but $R[Jt]$ satisfies the S_2 property, as a consequence of the previous theorem, we are still able to obtain information about the degree of $\mathrm{e}(I^n/J^n)$.

Corollary 3.13. *Let $(R, \mathfrak{m})$ be a formally equidimensional, analytically unramified local domain with infinite residue field and of positive dimension, and let $J \subseteq I$ be proper ideals of R with J equimultiple and $I \ne J$. Assume that $J = J_{\{1\}}$ (which holds, for instance, when $R[Jt]$ satisfies the S_2 property). Then*

$$\ell(J) - 1 \le \deg \mathrm{e}(I^n/J^n) \le \ell(J),$$

with equality on the left-hand side if and only if $J \subseteq I$ is a reduction.

It is worth comparing the above result with [5, Theorem 4.8] which states the following: if $J \subseteq I$ is a reduction, $J \neq I$, $R[Jt]$ is (S_2) and $\lambda(I/J) < \infty$, then $\ell(J) = \dim R$ and $\deg \lambda(I^n/J^n) = \dim R - 1$. As shown in the next two results, one can obtain the conclusion $\ell(J) = \dim R$ of [5, Theorem 4.8] without assuming that $J \subseteq I$ is a reduction.

Proposition 3.14. *Let $(R, \mathfrak{m})$ be a formally equidimensional local ring and $J \subseteq I$ proper ideals such that $\lambda(I/J) < \infty$. Assume that $\ell(J) < \dim R$. Then $I \subseteq J_{\{1\}}$.*

Proof. Consider the finitely generated $R[Jt]$-module $\mathcal{M} = \bigoplus_{n\geq 0} IJ^n/J^{n+1}$. Since $\lambda(I/J) < \infty$ we can choose k such that $\mathfrak{m}^k\mathcal{M} = 0$, so $\mathcal{M}$ is a finitely generated $R[Jt]/\mathfrak{m}^k R[Jt]$-module. But $\dim R[Jt]/\mathfrak{m}^k R[Jt] = \ell(J) < \dim R$, so $\dim \mathcal{M} < \dim R$, which by 3.9 (c) implies that $I \subseteq J_{\{1\}}$. □

As an immediate consequence we obtain the following.

Corollary 3.15. *Let $(R, \mathfrak{m})$ be a formally equidimensional local domain and $J \subseteq I$ proper ideals with $J \neq I$ such that $\lambda(I/J) < \infty$. Assume that $R[Jt]$ satisfies Serre's condition S_2. Then $\ell(J) = \dim R$.*

Proof. Since $R[Jt]$ is (S_2), so is $R[Jt, t^{-1}]$ ([2, Proposition 2.6]), and therefore $(J^n)_{\{1\}} = J^n$ for all n. If $\ell(J) < \dim R$, by the previous proposition we have $I \subseteq J_{\{1\}} = J$. □

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