FIRST COEFFICIENT IDEALS AND THE S_2-IFICATION OF A REES ALGEBRA

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ABSTRACT. Let $(A, \mathfrak{m}, k)$ be a d-dimensional ($d \geq 1$) quasi-unmixed analytically unramified local domain with infinite residue field. If I is an $\mathfrak{m}$-primary ideal, Shah defined the first coefficient ideal of I to be the largest ideal $I_{\{1\}}$ containing I such that $e_i(I) = e_i(I_{\{1\}})$ for $i = 0, 1$. Assume that A is (S_2) and let $\widetilde{S} = \oplus_{n\in\mathbb{Z}} I_n t^n$ be the S_2-ification of the extended Rees algebra $S = A[It, t^{-1}]$. We prove that $I_n = (I^n)_{\{1\}}$ for every $n \geq 1$. One of the consequences is a procedure of computing first coefficient ideals.

INTRODUCTION

Let $(A, \mathfrak{m}, k)$ be a quasi-unmixed local ring and let I be an $\mathfrak{m}$-primary ideal of A. Shah [15, Theorem 1] proved that there exists a unique ideal $I_{\{1\}}$, the first coefficient ideal of I, that is maximal among the ideals containing I for which the first two Hilbert coefficients are equal to those of I. The structure and properties of these first coefficient ideals have been extensively analyzed by Heinzer, Lantz, Johnston, and Shah ([15], [7], [8], [9]).

The purpose of this note is to find a procedure of computing first coefficient ideals. We do this by proving that for a large class of rings, $(I^n)_{\{1\}}$ is exactly the homogeneous part of degree n of the S_2-ification of the Rees algebra $R = A[It]$. This reduces the problem to the computation of the ring of endomorphisms of the canonical ideal of R. The construction of the S_2-ification of a Rees algebra has also been studied by Noh and Vasconcelos in [14].

In the first section we establish the notation and introduce the main concepts. Section 2 contains the main result of this note. Let $(A, \mathfrak{m}, k)$ be a d-dimensional ($d \geq 1$) quasi-unmixed, analytically unramified local domain with infinite residue field that satisfies the condition (S_2), and I an $\mathfrak{m}$-primary ideal. Then the homogeneous component of degree $n \geq 1$ of the S_2-ification of $S = A[It, t^{-1}]$ is equal to $(I^n)_{\{1\}}$. If $\operatorname{ht} I \geq 2$ a similar statement holds for the S_2-ification of $R = A[It]$. The last section contains some applications of this result and an explicit way to compute first coefficient ideals.

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1. Preliminaries

1.1. The (S_2) property of Serre. Let A be a Noetherian ring and let M be a finitely generated A-module. We say that M satisfies Serre's (S_2) property if for every prime ideal $\mathfrak{p}$ of A

$$\operatorname{depth} M_{\mathfrak{p}} \geq \inf\{2, \dim M_{\mathfrak{p}}\}. \quad (*)$$

We say that the ring A satisfies (S_2) if it satisfies (S_2) as an A-module (no embedded prime ideals and $\operatorname{ht}\mathfrak{p} = 1$ for all $\mathfrak{p} \in \operatorname{Ass}(A/xA)$ for any regular element $x \in A$). In the literature there is another definition for the (S_2) property where the condition $(*)$ is replaced by

$$\operatorname{depth} M_{\mathfrak{p}} \geq \inf\{2, \operatorname{ht}\mathfrak{p}\}. \quad (**)$$

In general $(**)$ implies $(*)$ and if M is a faithful A-module, $(*)$ is equivalent to $(**)$.

1.2. Remark. [6, (EGA) 5.7.11] Let $A \hookrightarrow B$ be a finite extension of rings. If B satisfies (S_2) as an A-module then B satisfies (S_2) as a ring (irrespective of the chosen definition). The reverse implication is also true if, for example, B is local and catenary (therefore equidimensional).

Following [10] we now define the S_2-ification of a Noetherian domain.

1.3. Definition. Let A be a Noetherian domain with quotient field $Q(A)$. We say that a domain B is an S_2-ification of A if:

(1) $A \subseteq B \subseteq Q(A)$ and B is module-finite over A,
(2) B is (S_2) as an A-module, and
(3) for all b in $B \setminus A$, $\operatorname{ht} D(b) \geq 2$ where $D(b) = \{a \in A \mid ab \in A\}$.

1.4. Remark. ([10, 2.4]) In general, the S_2-ification of a domain might not exist, but if there is one, then it must be unique. Denote

$$C := \{b \in Q(A) \mid \operatorname{ht} D(b) \geq 2\}.$$

Note that C can also be written $C = \bigcap \big\{A_{\mathfrak{p}} \mid \operatorname{ht}\mathfrak{p} = 1\big\}$.

Then A has an S_2-ification $\tilde{A}$ if and only if C is a finite extension of A, in which case $\tilde{A} = C$. It is also easy to observe that $\tilde{A}$ is a finite extension of A inside the quotient field, minimal with the property that it has the (S_2) property as an A-module. In many instances we will see it in this way.

1.5. Remark. If A is a universally catenary, analytically unramified domain, then A has an S_2-ification ([6, (EGA) 5.11.2]).

1.6. Remark. ([10, 2.7]) If $(A, \mathfrak{m}, k)$ is a local domain and ω is a canonical module for A, then $A \hookrightarrow \operatorname{Hom}_A(\omega, \omega)$ is an S_2-ification of A.

For more results about S_2-ification we refer to [1], [2], and [10].

1.7. Coefficient ideals. Let $(A, \mathfrak{m}, k)$ be a local ring and I an $\mathfrak{m}$-primary ideal. For sufficiently large values of n, $\lambda(R/I^n)$ is a polynomial $P_I(n)$ in n

of degree d, the Hilbert polynomial of I. We write this polynomial in terms of binomial coefficients:

$$P_I(n) = e_0(I)\binom{n+d-1}{d} - e_1(I)\binom{n+d-2}{d-1} + \cdots + (-1)^d e_d(I)$$

The coefficients $e_i(I)$ are integers and we call them the Hilbert coefficients of I.

In [15] the following theorem is proved.

1.8. Theorem (Shah)**.** *Let $(A, \mathfrak{m}, k)$ be a quasi-unmixed local ring of dimension $d > 0$ with infinite residue field and I an $\mathfrak{m}$-primary ideal. Then for each integer k in $\{0, 1, \dots, d\}$ there exists a unique largest ideal $I_{\{k\}}$ containing I such that*

$$e_i(I) = e_i(I_{\{k\}}) \quad \textit{for} \quad i = 0, 1, \dots, k.$$

The ideal $I_{\{k\}}$ is called the k-th coefficient ideal of I.

$I_{\{0\}}$ is the integral closure of I and if I contains a regular element, then $I_{\{d\}}$ is the Ratliff-Rush closure of I.

1.9. For an ideal I of a local domain A, the blowup $\mathcal{B}(I)$ of I is the *model*

$$\mathcal{B}(I) = \{A[I/x]_P \mid 0 \neq x \in I \text{ and } P \in \operatorname{Spec}(A[I/x])\}.$$

$\mathcal{B}(I)$ is the set of all local rings between A and $Q(A)$ minimal with respect to domination in which the extension of I is a principal ideal. For the basic facts on models we refer the reader to [18, Chapter VI].

In [7, 3.2] the first coefficient domain $\mathcal{D}_1$ of I is defined to be the intersection of the local domains on the blowup $\mathcal{B}(I)$ of dimension at most 1 in which the maximal ideal is minimal over the extension of I. By [8, Theorem 3.17] it follows that $I_{\{1\}} = I\mathcal{D}_1 \cap A$. It is also observed that since k is infinite one can choose an element a of I such that each of the local domains in $\mathcal{B}(I)$ with maximal ideal minimal over I is a localization of the same affine piece $A[I/a]$.

1.10. Remark. As noted in [8], the conclusion of [8, Theorem 3.17] makes sense even for not necessarily $\mathfrak{m}$-primary ideals, so one can consider $I_{\{1\}} = I\mathcal{D}_1 \cap A$ as the definition for the first coefficient ideal of I when I is not necessarily $\mathfrak{m}$-primary. Our results will be valid in this more general context.

2. The S_2-ification of a Rees algebra

Throughout this section $R = A[It]$ and $S = A[It, t^{-1}]$ will denote the Rees algebra and respectively the extended Rees algebra associated to the ideal I of a Noetherian domain A.

2.1. Lemma. *Let A be a Noetherian domain that has an S_2-ification $\widetilde{A}$. Then for any nonzero element $a \in A$*

$$a\Big(\bigcap\{A_{\mathfrak{p}} \mid \mathfrak{p} \textit{ minimal over } (a)\}\Big) \cap A = a\widetilde{A} \cap A.$$

Proof. Denote by $\mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_n$ the minimal prime ideals over the ideal generated by a. Let $x = ab \in A$ with $b \in \cap\{A_{\mathfrak{p}_i} \mid i = 1, \dots, n\}$. For each $i \in \{1, \dots, n\}$ there exist $r_i \in A$ and $s_i \notin \mathfrak{p}_i$ such that $b = r_i/s_i$. Since $(s_1, s_2, \dots, s_n) \not\subseteq \mathfrak{p}_1 \cup \mathfrak{p}_2 \cup \dots \cup \mathfrak{p}_n$ there exist $t_1, t_2, \dots, t_n$ such that

$$s := t_1 s_1 + t_2 s_2 + \dots + t_n s_n \notin \mathfrak{p}_1 \cup \mathfrak{p}_2 \cup \dots \cup \mathfrak{p}_n$$

and we can write

$$b = \frac{t_1 r_1 + t_2 r_2 + \dots + t_n r_n}{t_1 s_1 + t_2 s_2 + \dots + t_n s_n}.$$

Then $\operatorname{ht}(a, s) \geq 2$ and $(a, s)b \subseteq A$ which implies $b \in \widetilde{A}$. □

2.2. Remark. Let A be a Noetherian domain and let I be an ideal in A. If $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ are prime ideals of A, then

$$I\Big(\bigcap_{i=1}^{n} A_{\mathfrak{p}_i}\Big) = \bigcap_{i=1}^{n} (IA_{\mathfrak{p}_i}).$$

Let $a \in \cap_{i=1}^{n} IA_{\mathfrak{p}_i}$. For each $i \in \{1, \dots, n\}$ there exist $r_i \in I$ and $s_i \notin \mathfrak{p}_i$ such that $a = r_i/s_i$. As in 2.1 we can find $r \in I$ and $s \notin \cup_{i=1}^{n} \mathfrak{p}_i$ with $a = r/s$.

2.3. Assume that $S = A[It, t^{-1}]$ has an S_2-ification $\widetilde{S}$. Then $\widetilde{S}$ is a graded subring of $Q(A)[t, t^{-1}]$. We thank the referee for suggesting the following proof.

Given a prime ideal $\mathfrak{p}$ in S, let $\mathfrak{p}^*$ denote the prime ideal generated by the homogeneous elements of $\mathfrak{p}$. If $\mathfrak{p}$ is not homogeneous, then $\operatorname{ht}\mathfrak{p} = \operatorname{ht}\mathfrak{p}^* + 1$ [3, 1.5.8]. Consider the S-module

$$S' = \bigcap_{\operatorname{ht}\mathfrak{p}^* \leq 1} S_{\mathfrak{p}}.$$

Clearly $S' \subseteq \cap\{S_{\mathfrak{p}} \mid \operatorname{ht}\mathfrak{p} = 1\} = \widetilde{S}$. The subset $Z := \{\mathfrak{p} \in \operatorname{Spec} S \mid \operatorname{ht}\mathfrak{p}^* \geq 2\}$ of $\operatorname{Spec} S$ is closed under specialization and by [6, (EGA) 5.10.10] it follows that $\operatorname{depth} S'_{\mathfrak{p}} \geq 2$ when $\operatorname{ht}\mathfrak{p}^* \geq 2$. On the other hand, by [3, 1.5.9]

$$\operatorname{depth} S'_{\mathfrak{p}} = \operatorname{depth} S_{\mathfrak{p}} = \operatorname{depth} S_{\mathfrak{p}^*} + 1 \geq 2$$

if $\operatorname{ht}\mathfrak{p}^* = 1$, but $\mathfrak{p}$ is not homogeneous. So for any prime $\mathfrak{p}$ with $\operatorname{ht}\mathfrak{p} \geq 2$ we have $\operatorname{depth} S'_{\mathfrak{p}} \geq 2$, which means that S' satisfies the (S_2) property as an S-module, hence $S' = \widetilde{S}$.

For any open set $U \supseteq \{\mathfrak{p} \in \operatorname{Spec} S \mid \operatorname{ht}\mathfrak{p}^* \leq 1\}$ there exists a homogeneous ideal $\mathfrak{a} \subset S$ such that $\operatorname{ht}\mathfrak{a} \geq 2$ and $\{\mathfrak{p} \in \operatorname{Spec} S \mid \operatorname{ht}\mathfrak{p}^* \leq 1\} \subseteq D(\mathfrak{a}) \subseteq U$ where $D(\mathfrak{a}) = \{\mathfrak{p} \in \operatorname{Spec} S \mid \mathfrak{p} \not\supseteq \mathfrak{a}\}$. Indeed, if $U = D(\mathfrak{b})$ for some ideal $\mathfrak{b} \subset S$, we can take $\mathfrak{a} = \mathfrak{q}_1^* \cap \dots \mathfrak{q}_r^*$ where $\mathfrak{q}_1, \dots, \mathfrak{q}_r$ denote the minimal primes of $\mathfrak{b}$. By [6, (EGA) 5.9.3] it follows that

$$\widetilde{S} = \varinjlim_{\mathfrak{a}} \Gamma(D(\mathfrak{a}), \mathcal{O}_{D(\mathfrak{a})}) \quad (\mathfrak{a} \text{ homogeneous, } \operatorname{ht}\mathfrak{a} \geq 2)$$

where $\Gamma(D(\mathfrak{a}), \mathcal{O}_{D(\mathfrak{a})}) = \bigcap_{\mathfrak{p} \in D(\mathfrak{a})} S_{\mathfrak{p}}$.

But if $\mathfrak{a}$ is generated by the homogeneous elements $f_1, \dots, f_k \in S$, then

$$\Gamma(D(\mathfrak{a}), \mathcal{O}_{D(\mathfrak{a})}) = \bigcap_{i=1}^{k} S_{f_i},$$

which implies that $\Gamma(D(\mathfrak{a}), \mathcal{O}_{D(\mathfrak{a})})$ is a graded submodule of $Q(A)[t, t^{-1}]$. Then the same is true for $\widetilde{S}$.

Note that in the case when A has a canonical module one can also show that $\widetilde{S}$ is a graded submodule of $Q(A)[t, t^{-1}]$ by identifying $\widetilde{S}$ with the ring of endomorphisms of the (graded) canonical module of S (1.6).

So we can write $\widetilde{S} = \oplus_n I_n t^n$ with the components I_n contained in the quotient field of A. We now prove that if A has the (S_2) property, then these components are actually ideals in A. If A is integrally closed this is obvious since $\widetilde{S} \subseteq \overline{S} = \oplus \overline{I^n} t^n$ ($I^n = A$ for $n < 0$).

2.4. Lemma. *Let A be a Noetherian domain that has an S_2-ification $\widetilde{A}$. We also assume that $S = A[It, t^{-1}]$ has an S_2-ification $\widetilde{S} = \oplus_{n \in \mathbb{Z}} I_n t^n$ where $I_n \subseteq Q(A)$. Then $I_n \subseteq \widetilde{A}$ for all n.*

Proof. For $\mathfrak{a}$ an ideal in A we will denote $\mathfrak{a}' = \mathfrak{a}A[t, t^{-1}] \cap A[It, t^{-1}]$. Let $x \in I_n$. Then there exists a height 2 ideal J of $A[It, t^{-1}]$ such that $Jxt^n \subseteq A[It, t^{-1}]$. Consider $c(J)$ the ideal of A generated by the coefficients of all the elements of J. It is easy to see that $c(J)x \subseteq A$. We also have $JA[t, t^{-1}] \subseteq c(J)A[t, t^{-1}]$, so $J \subseteq c(J)A[t, t^{-1}] \cap A[It, t^{-1}] = c(J)'$ which implies $\operatorname{ht} c(J)' \geq 2$. Let $\mathfrak{p}$ be a minimal prime over $c(J)$ such that $\operatorname{ht} \mathfrak{p} = \operatorname{ht} c(J)$. Then

$$2 \leq \operatorname{ht} c(J)' \leq \operatorname{ht} \mathfrak{p}' = \operatorname{ht} \mathfrak{p} = \operatorname{ht} c(J)$$

where the equality $\operatorname{ht} \mathfrak{p}' = \operatorname{ht} \mathfrak{p}$ is a result proved in [13, page 121]. We now have $\operatorname{ht} c(J) \geq 2$ and $c(J)x \subseteq A$, implying that $x \in \tilde{A}$.

□

We now prove the main ingredient necessary to reduce the computation of the first coefficient ideals to the computation of the canonical ideal of the Rees algebra.

2.5. Theorem. *Let $(A, \mathfrak{m}, k)$ be a quasi-unmixed, analytically unramified local domain with infinite residue field and positive dimension. Let $\widetilde{S} = \oplus_{n \in \mathbb{Z}} I_n t^n$ be the S_2-ification of $S = A[It, t^{-1}]$. Then*

$$I_n \cap A = (I^n)_{\{1\}} \quad \text{for all } n \geq 1.$$

In particular, if A has the (S_2) property, then $I_n = (I^n)_{\{1\}}$ for all $n \geq 1$.

Proof. Choose an element a of I as in 1.9. To simplify the notation we will denote by $\Lambda_1(C)$ the set of height 1 prime ideals of a ring C. By the main theorem of [8] we have:

$$(I^n)_{\{1\}} = a^n \Big(\bigcap \big\{ A[I/a]_{\mathfrak{p}} | \mathfrak{p} \in \Lambda_1(A[I/a]), a \in \mathfrak{p} \big\} \Big) \cap A$$

One easily verifies that

$$A[I/a]_{\mathfrak{p}} = A[I/a][at,(at)^{-1}]_{\mathfrak{p}A[I/a][at,(at)^{-1}]} \cap Q(A) = (S_{at})_{\mathfrak{p}S_{at}} \cap Q(A).$$

Therefore we have

$$\begin{aligned}(I^n)_{\{1\}} &\supseteq a^n\big(\bigcap\big\{(S_{at})_{\mathfrak{q}} \cap Q(A) | \mathfrak{q} \in \Lambda_1(S_{at}), a \in \mathfrak{q}\big\}\big) \cap A \\ &\supseteq a^n\big(\big(\bigcap\big\{S_{\mathfrak{q}\cap S} | \mathfrak{q} \in \Lambda_1(S_{at}), a \in \mathfrak{q}\big\}\big)_{at} \cap Q(A)\big) \cap A \\ &\supseteq a^n\big(\big(\bigcap\big\{S_{\mathfrak{q}} | \mathfrak{q} \in \Lambda_1(S), a \in \mathfrak{q}\big\}\big)_{at} \cap Q(A)\big) \cap A.\end{aligned}$$

On the other hand,

$$\bigcap\big\{S_{\mathfrak{q}} | \mathfrak{q} \in \Lambda_1(S)\big\} = \widetilde{S} = \cdots + At^{-1} + A + I_1 t + I_2 t^2 + \cdots,$$

so

$$\begin{aligned}(I^n)_{\{1\}} &\supseteq a^n(\widetilde{S}_{at} \cap Q(A)) \cap A \\ &= a^n\Big(\Big(\bigcup_k \frac{I_k}{a^k}\Big)[at,(at)^{-1}] \cap Q(A)\Big) \cap A \\ &= a^n\Big(\bigcup_k \frac{I_k}{a^k}\Big) \cap A \supseteq I_n \cap A.\end{aligned}$$

To prove the other inclusion let us observe first that if $\mathfrak{q}$ is a minimal prime of $t^{-1}A[It,t^{-1}] = t^{-1}S$ then $S_{\mathfrak{q}} \cap Q(A)$ is a local domain whose maximal ideal is minimal over the extension of I. Moreover, every local domain in $\mathcal{B}(I)$ whose maximal ideal is minimal over the extension of I can be obtained in this way.

Using again [8, Theorem 3.17] we obtain

$$\begin{aligned}(I^n)_{\{1\}} &= I^n\big(\bigcap\big\{S_{\mathfrak{q}} | \mathfrak{q} \text{ minimal prime over } t^{-1}S\big\} \cap Q(A)\big) \cap A \\ &\subseteq t^{-n}\big(\bigcap\big\{S_{\mathfrak{q}} | \mathfrak{q} \text{ minimal prime over } t^{-1}S\big\}\big) \cap A.\end{aligned}$$

Using Lemma 2.1 we now get

$$(I^n)_{\{1\}} \subseteq t^{-n}\widetilde{S} \cap A = I_n \cap A$$

and the proof is finished. □

Shah [15, Theorem 4] has proved that $(I^n)_{\{1\}} = I^n$ for all n if and only if the associated graded ring $G_I(A)$ has no embedded prime ideals. By a result of Brumatti, Simis, and Vasconcelos [4, 1.5], if $\operatorname{ht} I \geq 2$ this is equivalent to the fact that $R = A[It]$ has the (S_2) property. Our statement is a generalization of Shah's result since $\widetilde{R}$ and $\widetilde{S}$ have the same homogeneous components in positive degree, an observation which is proved below.

2.6. Proposition. *Let $(A, \mathfrak{m}, k)$ be local domain with the (S_2) property, $R = A[It]$ and $S = A[It, t^{-1}]$. Assume that R and S have the S_2-ifications $\widetilde{R}$ and $\widetilde{S}$ respectively.*

i) *If* $\widetilde{R} = \oplus_{n\geq 0} I_n t^n$ *then* $\widetilde{S} \subseteq \oplus_{n\in\mathbb{Z}} I_n t^n$ *where* $I_n = A$ *for* $n < 0$.
ii) *If* $\operatorname{ht} I \geq 2$ *and* $\widetilde{S} = \oplus_{n\in\mathbb{Z}} I_n t^n$, *then* $\widetilde{R} \subseteq \oplus_{n\geq 0} I_n t^n$.

Consequently, if $\operatorname{ht} I \geq 2$, *then* $\widetilde{R}$ *and* $\widetilde{S}$ *have the same homogeneous components in positive degree.*

Proof. i) Let $\widetilde{S} = \oplus J_n t^n$ be the S_2-ification of S and take $ft^n \in \widetilde{S}$, $f \in A$ and $n > 0$. Then there exists a height 2 ideal J of $A[It, t^{-1}]$ such that $Jft^n \subseteq A[It, t^{-1}]$. Let J^+ be the ideal of $A[It]$ generated by the elements $g \in A[It]$ such that there exists $b_1, \dots, b_m$ with

$$\sum_{j=1}^{m} b_j t^{-j} + g \in J \subseteq A[It, t^{-1}].$$

Note that for any such generator g of J^+ we have $gft^n \in A[It]$, i.e.

$$J^+ ft^n \subseteq A[It].$$

Since $J \subseteq (t^{-1}) + J^+ A[It, t^{-1}]$ we have $\operatorname{ht}((t^{-1}) + J^+ A[It, t^{-1}]) \geq 2$, hence $\operatorname{ht} J^+ G_I(A) \geq 1$. The algebra $\widetilde{S} = \oplus J_n t^n$ is a finite extension of S so there exists k such that $J_{s+k} \subseteq I^s$ for all $s \geq 1$. Then $I^k A[It] ft^n \subseteq A[It]$, so

$$(J^+ + I^k A[It]) ft^n \subseteq A[It].$$

Since $\operatorname{ht} J^+ G_I(A) \geq 1$ we obtain that $\operatorname{ht}(J^+ + I^k A[It]) \geq 2$ and therefore $ft^n \in \widetilde{R}$.

ii) If $\widetilde{S} = \oplus I_n t^n$ is the S_2-ification of the extended Rees algebra then $\oplus_{n\geq 0} I_n / I_{n+1}$ has the (S_1) property. By a straightforward generalization to filtrations of [4, 1.5], $T := \oplus_{n\geq 0} I_n t^n$ has the (S_2) property and therefore $\widetilde{R} \subseteq \oplus_{n\geq 0} I_n t^n$. (Once one observes that if P is a prime ideal of $T = \oplus_{n\geq 0} I_n t^n$, then $P \supseteq It$ implies $P \supseteq \oplus_{n>0} I_n t^n$, basically since $I_n \subseteq \overline{I^n}$, the argument for the I-adic filtration given in the paper mentioned before can be followed word by word.) □

2.7. Remark. Note that the first part of Proposition 2.6 implies that if $A[It]$ is (S_2) then $A[It, t^{-1}]$ is (S_2) and therefore $G_I(A)$ is (S_1), which is the other implication of [4, 1.5].

2.8. Remark. If a Noetherian domain R has an S_2-ification $\widetilde{R}$, then for every multiplicatively closed subset T of R, $T^{-1}\widetilde{R}$ is an S_2-ification of $T^{-1}R$. Indeed, $T^{-1}\widetilde{R}$ is S_2 over $T^{-1}R$, module-finite, and for every element $s/t \in T^{-1}\widetilde{R}$, $D(s/t) = D(s/1) = T^{-1}D(s) \subseteq T^{-1}R$ has height at least 2.

If we apply this observation to the ring $A[It, t^{-1}]$ with $T = A \setminus \mathfrak{p}$ where $\mathfrak{p}$ is a prime ideal of A, by Theorem 2.5 we get that $(I_{\mathfrak{p}})_{\{1\}} = (I_{\{1\}})_{\mathfrak{p}}$.

2.9. Remark. It is known that if I is an ideal in A, then $x \in \overline{I}$ if and only if there exists a height 1 ideal $\mathfrak{a}$ of A such that $\mathfrak{a}x^m \subseteq I^m$ for all m. We obtain a similar conclusion for first coefficient ideals, but only in one direction.

Let $(A, \mathfrak{m}, k)$ be a quasi-unmixed, analytically unramified local domain with infinite residue field and I an ideal of height at least 2. Then there exists an ideal $\mathfrak{a}$ in A of height 2 such that

$$\mathfrak{a}(I^m)_{\{1\}} \subseteq I^m \quad \text{for all } m.$$

In particular, if $x \in I_{\{1\}}$, then there exists a height 2 ideal $\mathfrak{a}$ of A such that $\mathfrak{a}x^m \in I^m$ for all m.

Since $\widetilde{R}$ is a finite extension of $R = A[It]$ there exists a height 2 ideal J of R such that $J\widetilde{R} \subseteq R$. Let $x \in (I^m)_{\{1\}}$. Then $xt^m \in \widetilde{R}$, so $Jxt^m \subseteq R$. We now have $c(J)x \subseteq I^m$ and $\operatorname{ht} c(J) \geq 2$. Indeed, if there exists a height one prime ideal $\mathfrak{p}$ in A with $c(J) \subseteq \mathfrak{p}$ then $\operatorname{ht} \mathfrak{p}A[t] \cap A[It] = 1$ and

$$J \subseteq JA[t] \cap A[It] \subseteq c(J)A[t] \cap A[It] \subseteq \mathfrak{p}A[t] \cap A[It]$$

contradicting the fact that $\operatorname{ht} J \geq 2$.

However, the existence of a height 2 ideal $\mathfrak{a}$ such that $\mathfrak{a}x^m \subseteq I^m$ for all m does not imply that $x \in I_{\{1\}}$ as the following easy example shows. Let $A = k[x, y]$ and $I = (x^3, y^3)$. Then $R = A[It]$ is a Cohen-Macaulay ring and by Theorem 2.5 we have $I_{\{1\}} = I$. On the other hand, $(x, y)x^{2m}y^{2m} \subseteq I^m$ for all $m \geq 1$, but $x^2y^2 \notin I_{\{1\}} = (x^3, y^3)$.

If I is an ideal in a Noetherian ring we define I^{unm} to be the intersection of the primary components coresponding to the minimal prime ideals over I. Let us recall that an ideal I is called equimultiple if $\ell(I) = \operatorname{ht} I$ where $\ell(I) = \dim G_I(A)/\mathfrak{m}G_I(A)$ is the analytic spread of I.

In [14, 2.5], Noh and Vasconcelos proved that if A is a Cohen-Macaulay ring and I is an equimultiple ideal of positive height such that $A[It]$ has the (S_2) property, then all the powers I^n are unmixed ideals. Using a technique employed in [12] by Huneke (see also [11, exercise 10.11]) we will extend this result.

2.10. Proposition. *Let A be a quasi-unmixed, analytically unramified local domain with the (S_2) property and let I be an equimultiple ideal of height at least 2. If $\widetilde{R} = \oplus_{n\geq 0} I_n t^n$ is the S_2-ification of the Rees algebra $R = A[It]$, then*

$$(I^n)^{unm} \subseteq I_n.$$

Proof. Since $\ell(I) = \ell(I^n)$ and $I_n = (I^n)_{\{1\}}$ it follows that it is enough to prove the statement for $n = 1$. We may also assume that $\operatorname{ht} I < \dim A$, otherwise there is nothing to prove. Let $I = (J_1 \cap \cdots \cap J_k) \cap (J_{k+1} \cap \cdots \cap J_n)$ be an irredundant primary decomposition of I, where $\sqrt{J_i} = \mathfrak{p}_i$ and $\mathfrak{p}_1, \ldots, \mathfrak{p}_k$ are the minimal primes of I. Then $I^{unm} = J_1 \cap \cdots \cap J_k$ and to simplify the notation denote it by J.

Suppose there exists a prime $\mathfrak{p}$ in A such that $J_{\mathfrak{p}} \nsubseteq (I_{\{1\}})_{\mathfrak{p}}$ and choose one minimal with this property. Then localize at $\mathfrak{p}$ and in this way the problem reduces to the case $\operatorname{Supp}(J_{\{1\}}/I_{\{1\}}) = \{\mathfrak{m}\}$. Indeed, if $\mathfrak{q} \subset \mathfrak{p}, \mathfrak{q} \neq \mathfrak{p}$, then $I_{\mathfrak{q}} \subseteq J_{\mathfrak{q}} \subseteq (I_{\{1\}})_{\mathfrak{q}} = (I_{\mathfrak{q}})_{\{1\}}$ which by [8, Proposition 3.2] implies $(J_{\mathfrak{q}})_{\{1\}} = (I_{\mathfrak{q}})_{\{1\}}$. By Remark 2.8 we then get $(J_{\{1\}})_{\mathfrak{q}} = (I_{\{1\}})_{\mathfrak{q}}$. The ideal

I is still equimultiple and $\operatorname{ht} I < \dim A$ since the prime $\mathfrak{p}$ was not minimal over I ($J_{\mathfrak{p}_i} = I_{\mathfrak{p}_i}$). Since $J_{\{1\}}/I_{\{1\}}$ has finite length there exists n such that $\mathfrak{m}^n J_{\{1\}} \subseteq I_{\{1\}}$. Then for any $x \in J$ we have $\mathfrak{m}^n x \in I_{\{1\}}$, so $\mathfrak{m}^n xt \in \widetilde{R}$.

We have $\dim A > \ell(I) = \dim R/\mathfrak{m}R \geq \dim R/\mathfrak{m}\widetilde{R} \cap R = \dim \widetilde{R}/\mathfrak{m}\widetilde{R}$, thus $\operatorname{ht} \mathfrak{m}\widetilde{R} \geq 2$. But $\mathfrak{m}^n Rxt \subseteq \widetilde{R}$ which implies that $xt \in \widetilde{R}$, i.e. $x \in I_{\{1\}}$. □

2.11. Remark. Note that the assumption I equimultiple can be replaced by the condition

$$\ell(I_{\mathfrak{p}}) < \dim A_{\mathfrak{p}} \text{ for all primes } \mathfrak{p} \supseteq I, \mathfrak{p} \text{ not minimal over } I,$$

and the conclusion still holds.

3. Computing first coefficient ideals

The purpose of this section is to present a procedure to compute first coefficient ideals using the canonical ideal of the Rees algebra.

Let $A = k[X_1, X_2, \ldots, X_n] (n \geq 2)$ and $\mathfrak{m} = (X_1, \ldots, X_n)$. Then for an $\mathfrak{m}$-primary ideal I one can define $I_{\{1\}}$ in a similar way since $\lambda(A/I^n) = \lambda(A_{\mathfrak{m}}/I^n A_{\mathfrak{m}})$. As usual denote $R = A[It]$.

Let ω_R be the canonical ideal of R. Since ω_R is a monomial ideal (in t) of $R = A[It] \subseteq A[t]$ we can write

$$\omega_R = U_1 t^s R + U_2 t^{s+1} R + \cdots + U_k t^{s+k-1} R$$

where $U_1, U_2, \ldots, U_k$ are ideals in A.

In [9], following an idea of K. Smith, Heinzer and Lantz prove that if I is a monomial ideal in a polynomial ring, then $I_{\{1\}}$ is also a monomial ideal. Using Theorem 2.5 we are able to generalize this to any grading on the polynomial ring. The proof is obvious once one identifies $I_{\{1\}}$ with the homogeneous component of degree 1 of $\widetilde{R}$.

3.1. Proposition. *Let $A = k[X_1, X_2, \ldots, X_n]$ be a polynomial ring and I a graded ideal of A. Then $I_{\{1\}}$ is also a graded ideal.*

3.2. Proposition. *Let $A = k[X_1, X_2, \ldots, X_n]$ and I an ideal of A. For $i \in \{1, 2, \ldots, k\}$ denote*

$$V_i = (U_1 I^{m+i-1} + U_2 I^{m+i-2} + \cdots + U_{m+i}) : U_i$$

where $U_j = 0$ for $j > k$. Then for all m we have

$$(I^m)_{\{1\}} = V_1 \cap V_2 \cap \cdots \cap V_k.$$

Proof. By Theorem 2.5 and Proposition 2.6, $a \in (I^m)_{\{1\}}$ if and only if $at^m \in \widetilde{R}$. We also have

$$\widetilde{R} \cong \operatorname{Hom}_R(\omega_R, \omega_R) \cong (\omega_R :_{Q(R)} \omega_R).$$

But $at^m \in (\omega_R :_{Q(R)} \omega_R)$ if and only if

$$at^m U_i t^{s+i-1} \subseteq (U_1 t^s + U_2 t^{s+1} + \cdots + U_k t^{s+k-1}) A[It]$$

for all $i = 1, 2, \dots, k$, i.e.

$$aU_i \subseteq (U_1 I^{m+i-1} + U_2 I^{m+i-2} + \cdots + U_{m+i}) = V_i$$

for all $i = 1, 2, \dots, k$. □

For a procedure to compute the canonical ideal of an affine algebra we refer the reader to [17, Theorem 6.3.4].

3.3. Example. Let $A = k[x, y]$, $I = (x^8, x^3y^2, x^2y^4, y^8)$, and $R = A[It]$. Using the procedure mentioned above, a computation with Macaulay 2 shows that:

$$\begin{aligned} \omega_R &= (U_1 + U_4 t^3)R \subseteq R, \quad \text{where} \\ U_1 &= (x^6y^4, x^5y^6, x^4y^8, x^3y^{10}) \subseteq A \quad \text{and} \\ U_4 &= (x^{12}y^{16}, x^{11}y^{18}, x^{10}y^{20}, x^9y^{22}) \subseteq A. \end{aligned}$$

Then

$$\begin{aligned} I_{\{1\}} &= [(U_1 I) : U_1] \cap [(U_1 I^4 + U_4 I) : U_4] \\ &= (x^8, x^3y^2, x^2y^4, xy^6, y^8). \end{aligned}$$

Using one of the methods presented in [17] one could check that $A[I_{\{1\}}t]$ has the (S_2) property and therefore $\widetilde{R} = A[I_{\{1\}}t]$. Note that $\widetilde{R}$ is not a Cohen-Macaulay ring (depth $\widetilde{R} = 2$).

We now consider an ideal which is not primary for the maximal homogeneous ideal (using the definition adopted in 1.10):

3.4. Example. Let $A = k[x, y, z]$ and $I = (x^3, y^3, xyz)$. In this case

$$\begin{aligned} \omega_R &= (U_1 + U_2 t)R, \quad \text{where} \\ U_1 &= (yz, zx) \subseteq A \quad \text{and} \\ U_2 &= (xy^2z, x^2yz) \subseteq A. \end{aligned}$$

Then

$$\begin{aligned} I_{\{1\}} &= [(U_1 I + U_2) : U_1] \cap [(U_1 I^2 + U_2 I) : U_2] \\ &= (x^3, y^3, xyz, x^2y^2) \end{aligned}$$

In this case $A[I_{\{1\}}t]$ is a Cohen-Macaulay ring (it has depth 4), so $\widetilde{R} = A[I_{\{1\}}t]$.

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