WEAK NORMALIZATION IN GRADED EXTENSIONS AND WEAK SUBINTEGRAL CLOSURE OF IDEALS

CĂTĂLIN CIUPERCĂ

ABSTRACT. We study several aspects of the weak normalization in a graded extension of commutative rings. Applied to the Rees algebra of an ideal in a noetherian ring, the results obtained allow us to prove several properties of the weak subintegral closure of an ideal, a concept introduced by Vitulli and Leahy. The same methods are also used to eliminate one of the hypotheses in a theorem of Gaffney and Vitulli.

1. Introduction

The weak normalization of an abstract scheme was introduced by Andreotti and Bombieri [1] as a generalization of the weak normalization of a complex analytic variety defined earlier by Andreotti and Norguet [2]. About the same time, the related and more algebraic notion of seminormalization was introduced by Traverso [19]. In a geometric context, when passing from an algebraic variety X to its normalization $\overline{X}$, one point in the original variety may split into several points in the normalization, which may not be desirable in certain situations. This type of problem, also encountered by Andreotti and Norguet [2], is avoided if one passes instead to the weak normalization of X. This is an intermediate variety between X and its normalization which is obtained through a process that glues in $\overline{X}$ all the points that have the same image in X. The weak normalization of X satisfies universal mapping properties and has geometric and functorial properties that normalization usually lacks. From a more algebraic perspective, the properties and applications of the weak normalization and the closely related concept of seminormalization have been studied, among others, by Swan [18], Leahy and Vitulli [12, 20], Manaresi [14], and Yanagihara [23]. The survey paper of Vitulli

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[21] provides a comprehensive account of the work done in this area. More recently, the concept of seminormalization has been used in the minimal model program [11].

In this paper we look at weak normalization from an algebraic point of view that relies upon a characterization given by Manaresi [14] (see 1.2.1). Given an arbitrary graded extension of commutative rings $A \subseteq B$, we describe several properties of the weak normalization of A in B and its relation to the weak normalization in the induced extension of diagonal subalgebras $A^{\Delta} \subseteq B^{\Delta}$. We also establish several properties of the related notion of weak subintegral closure of an ideal, a concept introduced by Vitulli and Leahy [22]. From one point of view, the weak subintegral closure of the ideal I of the ring A is obtained from the weak normalization of the Rees algebra $A[It]$ the same way the integral closure $\overline{I}$ is obtained from the integral closure of $A[It]$. It is a closure operation on ideals whose relation to integral closure has been studied by Gaffney and Vitulli in [7]. With the different perspective provided by Manaresi's criterion, it is the main goal of this paper to better understand this closure operation and provide more insight into its relation with the integral closure of ideals as well as other closures related to it, particularly Epstein's notion of special integral closure [5] and the inner integral closure. We also hope that the general results regarding weak normalization in graded extensions provide useful insight into the behavior of weak normalization in both algebraic and geometric contexts.

We begin by recalling several concepts needed to build the definition of the weak subintegral closure of an ideal. Throughout, all rings are commutative and have an identity element.

Definition 1.1. Let $A \subseteq B$ be an integral extension of rings. The weak normalization ${}^*_BA$ and the seminormalization ${}^+_BA$ of A in B are defined by

$${}^*_BA = \{b \in B \mid \text{for every } \mathfrak{p} \in \operatorname{Spec}(A) \text{ there exists } n \in \mathbb{N} \text{ such that } (b/1)^{p^n} \in A_{\mathfrak{p}} + \operatorname{Rad}(B_{\mathfrak{p}})\}$$

and

$${}^+_BA = \{b \in B \mid (b/1) \in A_{\mathfrak{p}} + \operatorname{Rad}(B_{\mathfrak{p}}) \text{ for every } \mathfrak{p} \in \operatorname{Spec}(A)\},$$

where p is the characteristic exponent of the field $k(\mathfrak{p}) = A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$ and $\operatorname{Rad}(B_{\mathfrak{p}})$ is the Jacobson radical of $B_{\mathfrak{p}}$.

Different characterizations of ${}^*_BA$ and ${}^+_BA$ are obtained by considering *weakly subintegral* and *subintegral* extensions, respectively.

Definition 1.2 ([23])**.** An extension $A \subseteq B$ is said to be *weakly subintegral* if

(a) $A \subseteq B$ is integral;
(b) The induced map $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$ is bijective; and
(c) For every $\mathfrak{q} \in \operatorname{Spec}(B)$, the field extension $k(\mathfrak{q} \cap A) \to k(\mathfrak{q})$ is purely inseparable.

If in condition (c) one requires that the field extensions $k(\mathfrak{q} \cap A) \to k(\mathfrak{q})$ be isomorphisms, the extension is said to be *subintegral* ([18]).

The notions of weak normalization and seminormalization are essentially the notions of weak subintegral closure and subintegral closure, respectively. More precisely, for an integral extension $A \subseteq B$, it is known that ${}^*_BA$ is the largest weakly subintegral extension of A in B. Similarly, ${}^+_BA$ is the largest subintegral extension of A in B. If all the residue field extensions are separable (e.g., A contains a field of characteristic zero), the notions of subintegral and weakly subintegral extensions coincide, and ${}^+_BA$ coincides with ${}^*_BA$. We refer the reader to [13, 17, 18, 21–23] for extensive accounts of these notions.

In [14, Theorem I. 6] Manaresi proved the following characterization of weak normalization: If $A \subseteq B$ is an integral extension, then

$$ {}^*_BA = \{b \in B \mid b \otimes 1 - 1 \otimes b \text{ is nilpotent in } B \otimes_A B\}. \tag{1.2.1}$$

This description will play a crucial role in our approach to studying the weak normalization of a graded ring and the associated notion of weak subintegral closure of an ideal.

Remark 1.3. If $A \subseteq B$ is an extension of rings that is not necessarily integral, then one can define ${}^*_BA$ as ${}^*_{B'}A$, where $B' = \overline{A}^B$ is the integral closure of A in B. In this context, since every weakly subintegral extension of A in B is contained in $\overline{A}^B$, it is still true that ${}^*_BA$ is the largest weakly subintegral extension of A in B. Moreover, Manaresi's characterization of weak normalization becomes

$$ {}^*_BA = \{b \in \overline{A}^B \mid b \otimes 1 - 1 \otimes b \text{ is nilpotent in } \overline{A}^B \otimes_A \overline{A}^B\}. \tag{1.3.1}$$

Remark 1.4. The characterization (1.2.1) is only valid for integral extensions $A \subseteq B$. For arbitrary extensions we do need to consider only the elements $b \in \overline{A}^B$ as in (1.3.1). In general, $b \otimes 1 - 1 \otimes b$ nilpotent in $B \otimes_A B$ does not imply that b is integral over A. For example, consider the birational extension $A = k[x, xy, xz] \subseteq B = k[x, y, z]$ where k is a field. The ring A is isomorphic to a polynomial ring, hence it is normal. In $B \otimes_A B$ we have $xyz \otimes 1 = y \otimes xz = yx \otimes z = 1 \otimes xyz$, but xyz is not integral over A.

Reid, Roberts, and Singh [17] provided an elementwise equational characterization of weak subintegrality as follows. If $A \subseteq B$ is an extension of rings, an element $b \in B$ is said to be *weakly subintegral* over A if there exist $q \geq 0$ and $a_1, \ldots, a_{2q+1} \in A$ such that for every $n \in \{q+1, \ldots, 2q+1\}$ the element b satisfies the equations

$$b^n + \sum_{i=1}^{n} \binom{n}{i} a_i b^{n-i} = 0. \tag{1.4.1}$$

Note that every element $b \in B$ that is weakly subintegral over A must be integral over A. They proved that the extension $A \subseteq A[b]$ is weakly subintegral if and only if b is weakly subintegral over A. Therefore,

$$ {}^*_B A = \{b \in B \mid b \text{ satisfies equations of the type (1.4.1)}\}.$$

Based on this characterization, given a ring extension $A \subseteq B$ and an ideal I of A, Vitulli and Leahy [22, Definition 2.1] introduced the following notion of *weak subintegral closure* of I in B.

Definition 1.5. An element $b \in B$ is said to be weakly subintegral over I if there exist $q \geq 0$ and $a_i \in I^i$ $(1 \leq i \leq 2q+1)$ such that b satisfies the equations (1.4.1) for all $n \in \{q+1, \ldots, 2q+1\}$. The weak subintegral closure ${}^*_B I$ of I in B is the set of all elements in B that are weakly subintegral over I. For $B = A$, the notation ${}^*I$ will be used instead of ${}^*_A I$.

Vitulli and Leahy proved that ${}^*_B I$ is an ideal of ${}^*_B A$ [22, Proposition 2.11]. Moreover, for every $n \geq 1$, the ideal ${}^*_B(I^n)$ coincides with the homogeneous component of degree n of the weak normalization of the Rees algebra $A[It]$ in $B[t]$ [22, Theorem 3.3], a property that parallels a well known characterization of the integral closure $\overline{I^n}$. We do note, however, that the proofs from [22] of these properties are rather technical.

In this paper we revisit the concept of weak subintegral closure of an ideal from a different perspective. Instead of relying upon the elementwise characterization of weak normalization given by Reid, Roberts, and Singh [17], which inspired the definition of Vitulli and Leahy in the ideal case, we follow the characterization (1.3.1) of Manaresi and apply it directly to the Rees algebra of the ideal I.

The technical aspects involved when working with tensor products are detailed in Section 2 which also contains a study of the weak normalization in a graded extension of rings. If Σ is an additive abelian group, $\Delta = \mathbb{Z}\varepsilon \subseteq \Sigma$ is the subgroup generated by some $\varepsilon \in \Sigma$, and $A = \bigoplus_{i\in\Sigma} A_i$ is a Σ-graded ring, we denote by A^Δ the Σ-graded subring of A whose homogeneous components $[A^\Delta]_i$ vanish for $i \notin \Delta$ and coincide with A_i for $i \in \Delta$. We refer to A^Δ as the diagonal subalgebra of A along Δ. If $A = \bigoplus_{i\in\Sigma} A_i \subseteq B = \bigoplus_{i\in\Sigma} B_i$ is an extension of Σ-graded rings and Σ is torsion-free, one of the main consequences of the technical results from Section 2 shows that ${}^*_BA$ is a Σ-graded subring of B and $[{}^*_BA]_{n\varepsilon} = [{}^*_{B^\Delta}(A^\Delta)]_{n\varepsilon}$ for all $n \in \mathbb{Z}$ (Theorem 2.14). In particular, if $A \subseteq B$ is an extension of $\mathbb{N}$-graded rings which induces a graded extension of the d-th Veronese subrings $A^{(d)} \subseteq B^{(d)}$, then $[{}^*_BA]_{nd} = [{}^*_{B^{(d)}}A^{(d)}]_n$ for all $n \geq 0$. Applied to the Rees algebra $\mathcal{R}(I) = A[It]$ of an ideal I of an arbitrary ring A and the extension $A[It] \subseteq B[t]$ induced by a ring extension $A \subseteq B$, this implies that the homogeneous component I_d of ${}^*_{B[t]}A[It] = \bigoplus_{n\geq 0} I_n t^n$ coincides with the homogeneous component of degree one of the weak normalization of $\mathcal{R}(I^d)$ in $B[t]$ (Theorem 2.18). This recovers a consequence of [22, Theorem 3.3] of Vitulli and Leahy who showed that I_d coincides with ${}^*_B(I^d)$ (see Definition 1.5). By using Theorem 2.14 we also describe the weak normalization of the multi-Rees algebra associated to a finite family of ideals (Theorem 5.1). In Section 6 we show that extended Rees algebra $A[It, t^{-1}]$ and the Rees algebra $A[It]$ have the same homogeneous components in positive degrees.

In Section 3 we present a first application of looking at ${}^*I$ as

$${}^*I = \{x \in \overline{I} \mid xt \otimes 1 - 1 \otimes xt \text{ nilpotent in } \overline{A[It]} \otimes_{A[It]} \overline{A[It]}\},$$

where $\overline{A[It]}$ is the integral closure of $A[It]$ in $A[t]$. If $(A, \mathfrak{m})$ is a local ring with algebraically closed residue field $k = A/\mathfrak{m}$ and $\operatorname{char} k = 0$, and J is a minimal reduction of an $\mathfrak{m}$-primary ideal I, an interesting result of Gaffney and Vitulli [7, Theorem 4.6] shows that ${}^*J = I_> + J$, where $I_>$ is the ideal of A consisting of all the elements a such that $v(a) > v(I)$ for all the Rees valuations v of I. Using a technical result detailed in Lemma 3.1 and the tensor product characterization of ${}^*I$, we give a modified proof of their result that does not require that the residue field be algebraically closed.

Using the same characterization for ${}^*I$, in Section 4 we obtain several new results regarding the weak subintegral closure of an ideal. We show, for example, that if $A \subseteq B$ is a weakly subintegral extension and I is an ideal of A, then ${}^*(IB) \cap A = {}^*I$ (Theorem 4.2), mirroring a similar result that holds for integral extensions and integral closure of ideals. We also note that several other results already obtained by Vitulli and Leahy in [22] can be recovered with this approach.

2. Weak normalization in graded extensions

We begin with a few elementary remarks regarding tensor products and morphisms between spectra of rings.

2.1. If $A \subseteq B$ is an integral extension such that the induced map $f : \operatorname{Spec} B \to \operatorname{Spec} A$ is bijective and $\mathfrak{p}_1, \mathfrak{p}_2 \in \operatorname{Spec} A$ with $\mathfrak{p}_1 \subseteq \mathfrak{p}_2$, then $f^{-1}(\mathfrak{p}_1) \subseteq f^{-1}(\mathfrak{p}_2)$. Indeed, from the going-up property of integral extensions, there exists $\mathfrak{q} \in \operatorname{Spec} B$ with $\mathfrak{q} \supseteq f^{-1}(\mathfrak{p}_1)$ such that $f(\mathfrak{q}) = \mathfrak{p}_2$. Then $\mathfrak{q} = f^{-1}(\mathfrak{p}_2)$, and the conclusion follows.

2.2. Let $B \xrightarrow{\phi} C$ be a ring homomorphism such that the induced morphism $\operatorname{Spec} C \to \operatorname{Spec} B$ is surjective. If $\phi(b) = c$, then b is nilpotent if and only if c is nilpotent. In particular, $\operatorname{Ker} \phi$ is nilpotent.

2.3. If $B \subseteq C$ is an integral extension of A-algebras, then the kernel of the canonical ring homomorphism $B \otimes_A B \to C \otimes_A C$ is nilpotent. Indeed, the $\operatorname{Spec} A$-morphism of affine schemes $\operatorname{Spec} C \to \operatorname{Spec} B$ is surjective, hence the morphism $\operatorname{Spec} C \otimes_{\operatorname{Spec} A} \operatorname{Spec} C \to \operatorname{Spec} B \otimes_{\operatorname{Spec} A} \operatorname{Spec} B$ is surjective ([8, Chapitre I, Prop. 3.6.1]), i.e., $\operatorname{Spec}(C \otimes_A C) \to \operatorname{Spec}(B \otimes_A B)$ is surjective. The conclusion follows from (2.2).

2.4. If $A \subseteq B \subseteq C$ are ring extensions, the kernel of the canonical ring homomorphism $C \otimes_A C \to C \otimes_B C$ is generated by $\{b \otimes 1 - 1 \otimes b \mid b \in B\}$.

We now state a known lemma that gives an equational characterization of when a sum of simple tensors in a tensor product is equal to zero. For a proof we refer to [15, 16]. A slightly different version of this lemma can also be found in [3, Chapter I, §2.11].

Lemma 2.5. *Let R be a commutative ring and let M and N be R-modules. Assume that $\{m_\lambda\}_{\lambda\in\Lambda}$ and $\{n_\gamma\}_{\gamma\in\Gamma}$ are families of generators for M and N, respectively. Let $\{x_\lambda\}_{\lambda\in\Lambda}$ be a family with finite support of elements in N. Then $\sum_{\lambda\in\Lambda} m_\lambda \otimes x_\lambda = 0$ if and only if there exists a family $\{a_{\lambda\gamma}\}_{\lambda\in\Lambda,\gamma\in\Gamma}$ with finite support of elements in R such that*

$$x_\lambda = \sum_{\gamma\in\Gamma} a_{\lambda\gamma} n_\gamma \quad \textit{for all } \lambda \in \Lambda \textit{ and} \tag{2.5.1}$$

$$\sum_{\lambda\in\Lambda} m_\lambda a_{\lambda\gamma} = 0 \quad \textit{for all } \gamma \in \Gamma. \tag{2.5.2}$$

Remark 2.6. As one can check immediately, the direction "if" from the above Lemma holds for arbitrary families $\{m_\lambda\}_{\lambda\in\Lambda}$ and $\{n_\gamma\}_{\gamma\in\Gamma}$ of elements. For the "only if" direction however, the requirement that the elements $\{m_\lambda\}_{\lambda\in\Lambda}$ generate M is essential.

We will need a graded version of the above lemma, which we state and prove below for Σ-graded rings and modules, where Σ is an abelian group. Recall that if R is a Σ-graded commutative ring and M and N are Σ-graded R-modules, then the tensor product $M \otimes_R N$ is Σ-graded with homogeneous components $[M \otimes_R N]_\sigma = \{\sum_{i=1}^n m_i \otimes n_i \mid n \in \mathbb{N}, m_i \in M, n_i \in N, \deg m_i + \deg n_i = \sigma\}$ for $\sigma \in \Sigma$.

Lemma 2.7. *Let R be a Σ-graded commutative ring and let M and N be Σ-graded R-modules. Assume that $\{m_\lambda\}_{\lambda\in\Lambda}$ and $\{n_\gamma\}_{\gamma\in\Gamma}$ are families of non-zero homogeneous generators for M and N, respectively. Let $y = \sum_{\lambda\in\Lambda} m_\lambda \otimes x_\lambda \in [M \otimes_R N]_\sigma$ be a homogeneous element in the graded tensor product $M \otimes_A N$ where $\{x_\lambda\}_{\lambda\in\Lambda}$ is a family with finite support of homogeneous elements in N. Then $y = 0$ if and only if there exists a family $\{a_{\lambda\gamma}\}_{\lambda\in\Lambda,\gamma\in\Gamma}$ with finite support of homogeneous elements in R with $a_{\lambda\gamma} \in R_{\sigma-\deg m_\lambda-\deg n_\gamma}$ such that the equations* (2.5.1) *and* (2.5.2) *are satisfied.*

Proof. We only need to prove the "only if" direction. From Lemma 2.5 we know that there exists a family $\{a_{\lambda\gamma}\}_{\lambda\in\Lambda,\gamma\in\Gamma}$ with finite support of elements in R such that the equations (2.5.1) and (2.5.2) are satisfied. Let $a'_{\lambda\gamma}$ be the homogeneous component in $R_{\sigma-\deg m_\lambda-\deg n_\gamma}$ of $a_{\lambda\gamma}$. We will show that after replacing the elements $a_{\lambda\gamma}$ with $a'_{\lambda\gamma}$ the equations (2.5.1) and (2.5.2) still hold.

The sum $\sum_{\gamma\in\Gamma} a'_{\lambda\gamma}n_\gamma$ is the homogeneous component in $N_{\sigma-\deg m_\lambda}$ of the right hand side of each equation in (2.5.1). Since $x_\lambda \in N_{\sigma-\deg m_\lambda}$ we obtain

(2.7.1) $$x_\lambda = \sum_{\gamma\in\Gamma} a'_{\lambda\gamma}n_\gamma \quad \text{for all } \lambda\in\Lambda.$$

On the other hand, the sum $\sum_{\lambda\in\Lambda} m_\lambda a'_{\lambda\gamma}$ is the homogeneous component in $M_{\sigma-\deg n_\gamma}$ of the right hand side of each equation in (2.5.2), hence

(2.7.2) $$\sum_{\lambda\in\Lambda} m_\lambda a'_{\lambda\gamma} = 0 \quad \text{for all } \gamma\in\Gamma.$$

□

2.8 (Notation)**.** Let Σ be an additive abelian group and $\Delta \subseteq \Sigma$ a subgroup. If $A = \bigoplus_{i\in\Sigma} A_i$ is a Σ-graded ring, let $A^\Delta = \bigoplus_{i\in\Sigma}[A^\Delta]_i$ denote the graded subring of A with $[A^\Delta]_i = A_i$ for $i \in \Delta$ and $[A^\Delta]_i = 0$ for $i \in \Sigma \setminus \Delta$. When $A = \bigoplus_{i\geq 0} A_i$ is $\mathbb{N}$-graded and d is a positive integer, we denote by $A^{[d]}$ the graded subring $\oplus_{i\geq 0}A_{di}$ with the grading $[A^{[d]}]_{di} = A_{di}$ for all $i \geq 0$ and $[A^{[d]}]_i = 0$ for i not divisible by d. We also denote by $A^{(d)}$, and refer to it as the d-th Veronese subring, the same ring with the grading $[A^{(d)}]_i = A_{di}$ for all i.

Lemma 2.9. *Let $A = \bigoplus_{i\in\Sigma} A_i \subseteq B = \bigoplus_{i\in\Sigma} B_i$ be a Σ-graded extension of rings and $\Delta \subseteq \Sigma$ a subgroup. Then the canonical ring homomorphism*

$$B^\Delta \otimes_{A^\Delta} B^\Delta \xrightarrow{\psi_\Delta} A[B^\Delta] \otimes_A A[B^\Delta]$$

is injective.

Proof. As B^Δ is a ring, note that the ring $A[B^\Delta]$ coincides with the A-submodule of B generated by B^Δ. Let $\{b_\lambda\}_{\lambda\in\Lambda}$ be a family of non-zero homogeneous generators of B^Δ as an A^Δ-module. This family also generates $A[B^\Delta]$ as an A-module. If $\sigma \in \Delta$ and $y \in [\operatorname{Ker}\psi_\Delta]_\sigma$ is a homogeneous element, write $y = \sum_{\lambda\in\Lambda} b_\lambda \otimes x_\lambda$ where $\{x_\lambda\}_{\lambda\in\Lambda}$ is a family with finite support of homogeneous elements in B^Δ with $x_\lambda \in [B^\Delta]_{\sigma-\deg b_\lambda}$. Then $0 = \psi_\Delta(y) = \sum_{\lambda\in\Lambda} b_\lambda \otimes x_\lambda$ in $A[B^\Delta] \otimes_A A[B^\Delta]$. By Lemma 2.7 there exists a family $\{a_{\lambda\gamma}\}_{\lambda\in\Lambda,\gamma\in\Lambda}$ with finite support of homogeneous elements in A with $a_{\lambda\gamma} \in A_{\sigma-\deg b_\lambda - \deg b_\gamma}$ such that

$$x_\lambda = \sum_{\gamma\in\Lambda} a_{\lambda\gamma} b_\gamma \quad \text{for all } \lambda \in \Lambda \text{ and} \tag{2.9.1}$$

$$\sum_{\lambda\in\Lambda} b_\lambda a_{\lambda\gamma} = 0 \quad \text{for all } \gamma \in \Lambda. \tag{2.9.2}$$

Since $\sigma \in \Delta$ and $\deg b_\lambda \in \Delta$ for all $\lambda \in \Lambda$, we have $a_{\lambda\gamma} \in A^\Delta$ for all $\lambda, \gamma \in \Lambda$. Applying again Lemma 2.7 we obtain that $\sum_{\lambda\in\Lambda} b_\lambda \otimes x_\lambda = 0$ in $B^\Delta \otimes_{A^\Delta} B^\Delta$. $\square$

For $\Sigma = \mathbb{Z}$ and $\Delta = d\mathbb{Z}$, we record the following consequence for the Veronese subrings of graded rings with no components in negative degrees.

Corollary 2.10. *Let $A = \bigoplus_{i\geq 0} A_i \subseteq B = \bigoplus_{i\geq 0} B_i$ be a graded extension of rings. For $d \geq 1$, let $A^{(d)} = \bigoplus_{i\geq 0} A_{di}$ and $B^{(d)} = \bigoplus_{i\geq 0} B_{di}$ be the d-th Veronese subrings of A and B, respectively. Then the canonical ring homomorphism*

$$B^{(d)} \otimes_{A^{(d)}} B^{(d)} \xrightarrow{\psi_d} A[B^{(d)}] \otimes_A A[B^{(d)}]$$

is injective.

For an abelian group Σ and a subgroup $\Delta \subseteq \Sigma$ we also have the following.

Lemma 2.11. *Let $A \subseteq B$ be an integral graded extension of Σ-graded rings. Then the kernel of the canonical ring homomorphism*

$$B^\Delta \otimes_{A^\Delta} B^\Delta \xrightarrow{\varphi_\Delta} B \otimes_A B$$

is nilpotent.

Proof. The homomorphism φ_Δ factors as

$$B^\Delta \otimes_{A^\Delta} B^\Delta \xrightarrow{\psi_\Delta} A[B^\Delta] \otimes_A A[B^\Delta] \to B \otimes_A B,$$

where ψ_Δ is the injective map from Lemma 2.9 and the other map is induced by the integral extension $A[B^\Delta] \subseteq B$. By (2.3), the kernel of $A[B^\Delta] \otimes_A A[B^\Delta] \to B \otimes_A B$ is nilpotent. Then $\operatorname{Ker} \varphi_\Delta$ is nilpotent as well. $\square$

For Veronese subrings the extension $A[B^{(d)}] \subseteq B$ is integral because B is integral over $B^{(d)}$ and we do not need to assume that the extension $A \subseteq B$ is integral.

Corollary 2.12. *Let $A \subseteq B$ be an $\mathbb{N}$-graded extension rings and let $\varphi_d : B^{(d)} \otimes_{A^{(d)}} B^{(d)} \to B \otimes_A B$ be the canonical graded ring homomorphism. Then* $\operatorname{Ker} \varphi_d$ *is nilpotent.*

Remark 2.13. Let $A \subseteq B$ be a Σ-graded extension of rings. Also, assume that Σ is torsion-free if the extension is not integral. Then ${}^*_B A$ is a Σ-graded subring of B. Indeed, since Σ is torsion-free, the integral closure $\overline{A}^B$ of A in B is Σ-graded. (See [10, Sec. 2.5, 2.22] and also [10, Example 2.3.3] for an example where this fails when Σ is not torsion-free.)

From (1.3.1) we have ${}^*_B A = \{b \in \overline{A}^B \mid b \otimes 1 - 1 \otimes b \text{ is nilpotent in } \overline{A}^B \otimes_A \overline{A}^B\} = \{b \in \overline{A}^B \mid f(b) = g(b)\}$ where $f, g : \overline{A}^B \to (\overline{A}^B \otimes_A \overline{A}^B)_{\text{red}}$ are the graded ring homomorphisms given by $f(b) = b \otimes 1$ and $g(b) = 1 \otimes b$, respectively. If $F \subseteq \Sigma$ is a finite subset, $b = \sum_{\sigma \in F} b_\sigma \in \overline{A}^B$ with $b_\sigma \in B_\sigma$, and $f(b) = g(b)$, then $f(b_\sigma) = g(b_\sigma) \in [(\overline{A}^B \otimes_A \overline{A}^B)_{\text{red}}]_\sigma$, so $b_\sigma \in {}^*_B A$ for all $\sigma \in F$. Therefore ${}^*_B A$ is a Σ-graded subring of B. Using a different approach, for $\Sigma = \mathbb{Z}$, this was also proved in [22, Proposition 3.1].

We obtain the following result regarding the homogeneous components of the weak normalization in a graded ring extension.

Theorem 2.14. *Let $A \subseteq B$ be an Σ-graded extension of rings. Let $\varepsilon \in \Sigma$ and $\Delta = \mathbb{Z}\varepsilon$. Additionally, assume that Σ is torsion-free if the extension is not integral. Then $[{}^*_B A]_{n\varepsilon} = [{}^*_{B^\Delta}(A^\Delta)]_{n\varepsilon}$ for all $n \in \mathbb{Z}$, i.e., $({}^*_B A)^\Delta = {}^*_{(B^\Delta)}(A^\Delta)$.*

Proof. As noted in Remark 2.13, the integral closure $\overline{A}^B$ of A in B is a Σ-graded subring of B. Using equations of integral dependence one can see that $\overline{(A^\Delta)}^{B^\Delta} = \left(\overline{A}^B\right)^\Delta$, and hence, by Remark 1.3, we may replace B with $\overline{A}^B$ and assume that $A \subseteq B$ is an integral extension.

For $b \in [{}^*_B A]_{n\varepsilon} \subseteq B_{n\varepsilon}$, there exists k such that $(b \otimes 1 - 1 \otimes b)^k = 0$ in $B \otimes_A B$, so $(b \otimes 1 - 1 \otimes b)^k \in \operatorname{Ker} \varphi_\Delta$, where φ_Δ is the homomorphism from Lemma 2.11. It follows that $b \otimes 1 - 1 \otimes b$ is a nilpotent element in $B^\Delta \otimes_{A^\Delta} B^\Delta$, hence $b \in [{}^*_{B^\Delta} A^\Delta]_{n\varepsilon}$. For the opposite inclusion, it is clear that if $b \otimes 1 - 1 \otimes b$ is nilpotent in $B^\Delta \otimes_{A^\Delta} B^\Delta$, then $b \otimes 1 - 1 \otimes b \in B \otimes_A B$ is also nilpotent, showing that $[{}^*_{B^\Delta} A^\Delta]_{n\varepsilon} \subseteq [{}^*_B A]_{n\varepsilon}$. $\square$

With the notational conventions introduced in 2.8, in the case of $\mathbb{N}$-graded rings we have the following.

Corollary 2.15. *Let $A \subseteq B$ be an $\mathbb{N}$-graded extension of rings. Then*

$$[{}^*_B A]_{nd} = [{}^*_{B^{[d]}} A^{[d]}]_{nd} = [{}^*_{B^{(d)}} A^{(d)}]_n$$

for all n.

Remark 2.16. If A is a reduced $\mathbb{Z}$-graded ring, a similar result for the Veronese subrings of the seminormalization ${}^+A$ of A in its integral closure $\overline{A}$ was proved by Leahy and Vitulli [13, Proposition 2.7]. Their proof relies upon Hamann's criterion for seminormality and an adaptation of a construction of Swan to describe ${}^+A$.

For an arbitrary extension of rings $A \subseteq B$ and an ideal I of A we apply Theorem 2.14 to the induced t-graded extension $A[It] \subseteq B[t]$.

We begin with the following remark that applies to the case when $I = A$.

Remark 2.17. If $A \subseteq B$ is an extension of rings, then ${}^*_{B[t]}A[t] = ({}^*_B A)[t]$. In particular, if $A \subseteq B$ is a weakly subintegral extension (i.e., $B = {}^*_B A$), then $A[t] \subseteq B[t]$ is weakly subintegral. Indeed, if we denote $B' = \overline{A}^B$, it is known that $\overline{A[t]}^{B[t]} = B'[t]$. From the canonical isomorphism $B'[t] \otimes_{A[t]} B'[t] \cong (B' \otimes_A B')[t]$ that maps $bt^n \otimes ct^m$ to $(b \otimes c)t^{n+m}$ it follows that $at^n \otimes 1 - 1 \otimes at^n$ is nilpotent in $B'[t] \otimes_{A[t]} B'[t]$ if and only if $(a \otimes 1 - 1 \otimes a)t^n$ is nilpotent in $(B' \otimes_A B')[t]$. By (1.3.1), the conclusion follows.

Theorem 2.18. *Let $A \subseteq B$ be an extension of rings. For every ideal I of A, let $\mathcal{R}(I) = A[It]$ and let*

$${}^*_{B[t]}\mathcal{R}(I) = \oplus_{n \geq 0} I_n t^n \subseteq B[t]$$

be the weak normalization of $\mathcal{R}(I)$ in $B[t]$. Then I_d coincides with the homogeneous component of degree one of ${}^*_{B[t]}\mathcal{R}(I^d)$ for all $d \geq 1$. Moreover, I_d is an ideal of ${}^*_B A$ for every $d \geq 1$.

Proof. By Corollary 2.15, $[{}^*_{B[t]}\mathcal{R}(I)]_d = [{}^*_{B[t]^{[d]}}\mathcal{R}(I)^{[d]}]_d$, so I_d coincides with the homogeneous component of degree d of the weak normalization of $\mathcal{R}(I)^{[d]} = \oplus_{n \geq 0} I^{nd} t^{nd}$ in $B[t]^{[d]} =$

$\oplus_{n\geq 0}Bt^{nd}$. By substituting t for t^d, this means that I_d coincides with the homogeneous component of degree one of the weak normalization of $\oplus_{n\geq 0}I^{nd}t^n = \mathcal{R}(I^d)$ in $\oplus_{n\geq 0}Bt^n = B[t]$.

We also have ${}^*_{B[t]}\mathcal{R}(I) \subseteq {}^*_{B[t]}A[t] = \oplus_{n\geq 0}({}^*_BA)t^n$ by Remark 2.17. Moreover, $I_0 = {}^*_BA$. From the graded structure of ${}^*_{B[t]}\mathcal{R}(I)$ it now follows immediately that I_d is an ideal of ${}^*_BA$. □

Remark 2.19. As we will see later in Section 6, if we consider the weak normalization of the extended Rees algebra $A[It, t^{-1}]$ in $B[t, t^{-1}]$, then we obtain the same homogeneous components in non-negative degrees.

Remark 2.20. It was proved by Vitulli and Leahy [22, Theorem 3.3] that the ideal I_1, as defined in Theorem 2.18, coincides with the ideal consisting of all the elements $b \in B$ such that there exist $q \geq 0$ and $a_i \in I^i$ $(1 \leq i \leq 2q+1)$ such that b satisfies the equations $b^n + \sum_{i=1}^{n}\binom{n}{i}a_ib^{n-i} = 0$ for every $n \in \{q+1, \ldots, 2q+1\}$, an ideal which they called the weak subintegral closure of I in B and denoted ${}^*_BI$ (Definition 1.5). We subsequently adopt the same terminology and notation ${}^*_BI$ for I_1, the homogeneous component of degree one of the weak normalization of $A[It]$ in $B[t]$, and therefore

$${}^*_BI = \{x \mid xt\otimes 1 - 1\otimes xt \text{ is nilpotent in } \mathcal{S}(I)\otimes_{A[It]}\mathcal{S}(I)\},$$

where $\mathcal{S}(I)$ is the integral closure of $\mathcal{R}(I)$ in $B[t]$.

For $B = A$ we obtain the weak subintegral closure ${}^*I = {}^*_AI$ of I (in A). In fact, as the following Corollary shows, we can always reduce to this case, as ${}^*_BI$ coincides with ${}^*(IC)$ where $C = {}^*_BA$.

Corollary 2.21. *Let $A \subseteq B$ be an extension of rings and I an ideal of A. Let $C = {}^*_BA$. Then:*

(a) ${}^*_{B[t]}(A[It]) = {}^*_{C[t]}(C[(IC)t])$;

(b) ${}^*_BI = {}^*(IC)$.

Proof. (a) We begin with the following observation: If $A \subseteq B \subseteq D$ are ring extensions, $x \in D$, and $A \subseteq B$ is weakly subintegral, then $A[x] \subseteq B[x]$ is weakly subintegral, too. To see this, let J be the kernel of the natural surjection $B[X] \to B[x]$. By Remark 2.17, the extension $A[X] \to B[X]$ is weakly subintegral. Then so is the extension $A[X]/(J \cap A[X]) \subseteq B[X]/J$, i.e., $A[x] \subseteq B[x]$ is weakly subintegral.

From this observation it follows that the extension $A[It] \subseteq C[(IC)t]$ is weakly subintegral, which implies that ${}^*_{B[t]}(A[It]) = {}^*_{B[t]}(C[(IC)t])$. On the other hand, ${}^*_{B[t]}(C[(IC)t]) \subseteq {}^*_{B[t]}(C[t]) = ({}^*_B C)[t] = C[t]$, hence ${}^*_{B[t]}(C[(IC)t]) = {}^*_{C[t]}(C[(IC)t])$, which establishes (a).

Part (b) follows from (a) and Theorem 2.18. □

3. Inner and special integral closures

For an ideal I in a noetherian ring A and an element $a \in A$, we denote $\operatorname{ord}_I(a) = \sup\{n \mid a \in I^n\} \in \mathbb{N} \cup \{\infty\}$. The limit $\overline{v}_I(a) := \lim_{n\to\infty} \operatorname{ord}(a^n)/n$ always exists (possibly infinite), and defines the so-called *asymptotic Samuel function* of the ideal I. Moreover, $\overline{v}_I(a) \geq k$ if and only if $a \in \overline{I^k}$. We refer the reader to [10, Section 6.9 and Chapter 10] for a detailed discussion of the properties of this function.

In [7, Section 4] Gaffney and Vitulli considered the ideal $I_>$ consisting of all the elements $a \in A$ such that $\overline{v}_I(a) > 1$ and studied its relation to the weak subintegral closure ${}^*I$ of I in A. The ideal $I_>$ is a particular case of the ideals $I_{>\alpha}$ ($\alpha \in \mathbb{Q}$) discussed in [10, Section 10.5] where $I_> = I_{>1}$. It is also a special case of the J-special integral closure $I^{J-\mathrm{sp}}$ of I defined in [6, Section 5] where $I_>$ is recovered for $J = I$ and is referred to as the inner integral closure of I. For equivalent descriptions of $I_>$ we refer the reader to [5, Proposition 5.3] and [6, Theorem 5.4]. Another case of the J-special integral closure that is of interest to us is obtained in a local ring $(A, \mathfrak{m})$ by taking $J = \mathfrak{m}$. This is the so-called special part of the integral closure $I^{-\mathrm{sp}}$, as defined in [5, Section 5], and is the ideal consisting of all the elements $x \in A$ such that $x^n \in \overline{\mathfrak{m}I^n}$ for some n. As noted in [5, Proposition 5.3], if I is $\mathfrak{m}$-primary, then $I_> = I^{-\mathrm{sp}}$.

Using the equational definition of ${}^*I$ (Definition 1.5), Gaffney and Vitulli [7, Proposition 4.4] observed that $I_> \subseteq {}^*I$ for every ideal I in a noetherian ring. (This inclusion can also be obtained by using the tensor product characterization of ${}^*I$; see Remark 4.9.) Moreover, Gaffney and Vitulli proved that if $(A, \mathfrak{m}, k)$ is a local ring such that the residue field k is algebraically closed of characteristic zero and J is a minimal reduction of an $\mathfrak{m}$-primary ideal

I, then $J + I_{>} = {}^{*}J$ ([7, Theorem 4.6]). Note that this is in fact a statement about ideals generated systems of parameters, for it is equivalent to saying that $J + J_{>} = {}^{*}J$ for every $\mathfrak{m}$-primary parameter ideal J. Indeed, J is a minimal reduction of an $\mathfrak{m}$-primary ideal if and only if J is generated by a full system of parameters, and since $J \subseteq I$ is a reduction we have $J_{>} = I_{>}$.

Gaffney and Vitulli also noted that their proof required that the residue field be algebraically closed ([7, p. 2105]). By using the characterization ${}^{*}I = \{x \in \overline{I} \mid xt \otimes 1 - 1 \otimes xt$ nilpotent in $\overline{A[It]} \otimes_{A[It]} \overline{A[It]}$, where $\overline{A[It]}$ is the integral closure of $A[It]$ in $A[t]$, we are able to give a modified proof of their result [7, Theorem 4.6] that eliminates this restriction on the residue field. (This also eliminates the same restriction from several consequences of this theorem that were discussed in [7, §4].)

We begin with a technical lemma that is essential for establishing the result when the residue field is not necessarily algebraically closed.

Lemma 3.1. *Let k be a field with $\operatorname{char} k = 0$ and let $F(\underline{X}, T) = T^s + F_1(\underline{X})T^{s-1} + \cdots + F_s(\underline{X}) \in k[X_1, \ldots, X_d, T]$ be an irreducible homogeneous polynomial of degree $s \geq 1$, where $F_i(\underline{X}) = F_i(X_1, \ldots, X_d)$ is homogeneous of degree i. If there exists $n \geq 1$ such that*

$$(T_1 - T_2)^n \in \langle F(\underline{X}, T_1), F(\underline{X}, T_2)\rangle \subseteq k[X_1, \ldots, X_d, T_1, T_2],$$

then $s = 1$.

Proof. Let $R = k[X_1, \ldots, X_d]$. If $Q(R)$ denotes the quotient field of R, the polynomial $F(\underline{X}, T) \in R[T]$ is also irreducible in $Q(R)[T]$, hence $Q(R)[T]/\langle F(\underline{X}, T)\rangle$ is an integral domain. As $\operatorname{char} Q(R) = 0$, this implies that the ring

$$Q(R)[T_1]/\langle F(\underline{X}, T_1)\rangle \otimes_{Q(R)} Q(R)[T_2]/\langle F(\underline{X}, T_2)\rangle \cong Q(R)[T_1, T_2]/\langle F(\underline{X}, T_1), F(\underline{X}, T_2)\rangle$$

is reduced ([4, Chapter 5, §15]), so $T_1 - T_2 \in \langle F(\underline{X}, T_1), F(\underline{X}, T_2)\rangle Q(R)[T_1, T_2]$. Then there exist $h(\underline{X}) \in R \setminus \{0\}$ and $g_1(\underline{X}, T_1, T_2), g_2(\underline{X}, T_1, T_2) \in R[T_1, T_2]$ such that

$$h(\underline{X})(T_1 - T_2) = g_1(\underline{X}, T_1, T_2)F(\underline{X}, T_1) + g_2(\underline{X}, T_1, T_2)F(\underline{X}, T_2). \tag{3.1.1}$$

The field k is infinite and $h(\underline{X}) \neq 0$, so there exist $\alpha_1, \ldots, \alpha_d \in k$ such that $h(\alpha_1, \ldots, \alpha_d) \neq 0$. Let $\overline{k}$ be an algebraic closure of k and choose $\beta \in \overline{k}$ such that $F(\alpha_1, \ldots, \alpha_d, \beta) = 0$. If we set $X_i = \alpha_i$ $(1 \leq i \leq d)$ and $T_2 = \beta$ in (3.1.1), in the ring $\overline{k}[T_1]$ we have

$$h(\alpha_1, \ldots, \alpha_d)(T_1 - \beta) = g_1(\alpha_1, \ldots, \alpha_d, T_1, \beta)F(\alpha_1, \ldots, \alpha_d, T_1).$$

Since $h(\alpha_1, \ldots, \alpha_d) \neq 0$, the degree in T_1 on the right hand side must be one. On the other hand, $F(\alpha_1, \ldots, \alpha_d, T_1)$ is a monic polynomial of degree s, so $s = 1$. □

We are now able to prove the theorem of Gaffney and Vitulli without assuming that the residue field is integrally closed. We also state a version of the theorem for arbitrary ideals that involves the special part of the integral closure. If I is $\mathfrak{m}$-primary, then $I_> = I^{-\mathrm{sp}}$ and one recovers the original conclusion $J + I_> =^* J$.

Theorem 3.2. *Let $(A, \mathfrak{m}, k)$ be a local noetherian ring such that $\operatorname{char} k = 0$. Let I be an ideal of A and let J be a minimal reduction of I. Then*

$$J + I_> \subseteq {}^*J \subseteq J + I^{-\mathrm{sp}}.$$

Proof. The inclusion $I_> = J_> \subseteq {}^*J$ was proved in [7, Proposition 4.4] for an arbitrary ideal in a noetherian ring, not necessarily local. See also Remark 4.9 for a different approach.

For the second inclusion, we begin the proof exactly as in [7, Theorem 4.6] by following the standard argument of considering the fiber cone of the ideal I and its Noether normalization given by the minimal generators of the minimal reduction J. Let $x_1, \ldots, x_l$ be a minimal set of generators of J, where $l = \ell(I)$ is the analytic spread of I. Let $y \in {}^*J$, $K := J + yA$, and consider the integral extension of Rees algebras $A[Jt] \subseteq A[Kt]$.

Since J is a minimal reduction, the fiber ring $F(J) = A[Jt]/\mathfrak{m}A[Jt]$ is a polynomial ring $k[X_1, \ldots, X_l]$ where X_i is the image of x_i in $J/\mathfrak{m}J$. In particular, $\mathfrak{m}A[Jt]$ is a prime ideal in $A[Jt]$. Since $y \in {}^*J$, $A[Kt]$ is contained in the weak normalization of $A[Jt]$ in $A[t]$, so there exists a unique prime ideal $\mathfrak{q} \in \operatorname{Spec} A[Kt]$ with $\mathfrak{q} \cap A[Jt] = \mathfrak{m}A[Jt]$. Moreover, $\mathfrak{q}$ is the unique minimal prime over $\mathfrak{m}A[Kt]$. Indeed, if $\mathfrak{q}'$ is any minimal prime over $\mathfrak{m}A[Kt]$, then $\mathfrak{q}' \cap A[Jt] \supseteq \mathfrak{m}A[Jt] = \mathfrak{q} \cap A[Jt]$. By (2.1) we have $\mathfrak{q}' \supseteq \mathfrak{q}$, and therefore $\mathfrak{q}' = \mathfrak{q}$.

The integral extension $A[Jt] \subseteq A[Kt]$ induces an integral extension of integral domains

$$k[X_1, \dots, X_l] = F(J) \hookrightarrow F(K)/\mathfrak{q}F(K) = k[X_1, \dots, X_l, \bar{y}] \cong k[X_1, \dots, X_l, T]/\langle F(\underline{X}, T)\rangle,$$

where $F(\underline{X}, T)$ is an irreducible homogeneous polynomial in $k[X_1, \dots, X_l, T]$ that is monic in T. This follows from the fact that y satisfies an equation of integral dependence over $(x_1, \dots, x_l) = J$.

At this point we deviate from the original proof of Gaffney and Vitulli. The condition $y \in {}^*J$ means that $yt \otimes 1 - 1 \otimes yt$ is nilpotent in $\overline{A[Jt]} \otimes_{A[Jt]} \overline{A[Jt]}$, where $\overline{A[Jt]}$ is the integral closure of $A[Jt]$ in $A[t]$. As $A[Jt] \subseteq A[Kt] \subseteq \overline{A[Jt]}$, by (2.3) it follows that $yt \otimes 1 - 1 \otimes yt$ is nilpotent in $A[Kt] \otimes_{A[Jt]} A[Kt]$, too.

Then the image of $yt \otimes 1 - 1 \otimes yt$ in

$$[F(K)/\mathfrak{q}F(K)] \otimes_{F(J)} [F(K)/\mathfrak{q}F(K)] \cong k[X_1, \dots, X_l, T_1, T_2]/\langle F(\underline{X}, T_1), F(\underline{X}, T_2)\rangle$$

is also nilpotent, which means that $T_1 - T_2$ is nilpotent in the ring

$$k[X_1, \dots, X_l, T_1, T_2]/\langle F(\underline{X}, T_1), F(\underline{X}, T_2)\rangle.$$

By Lemma 3.1, this implies that the degree of $F(\underline{X}, T)$ in T is one, so there exist $a_1, \dots, a_l \in A$ such that $\bar{y} - (\bar{a}_1\bar{x}_1 + \cdots + \bar{a}_l\bar{x}_l) \in \mathfrak{q}F(K)$. Since $\mathfrak{q}F(K)$ is the unique minimal prime of $F(K)$, the element $\bar{y} - (\bar{a}_1\bar{x}_1 + \cdots + \bar{a}_l\bar{x}_l)$ is nilpotent in $F(K)$, so $[y - (a_1x_1 + \cdots + a_lx_l)]^n \in \mathfrak{m}K^n$ for some n. This means that $y - (a_1x_1 + \cdots + a_lx_l) \in K^{-\mathrm{sp}}$ and therefore $y \in J + K^{-\mathrm{sp}} \subseteq J + I^{-\mathrm{sp}}$. $\square$

4. Properties of the weak subintegral closure of an ideal

If $A \subseteq B$ is an integral extension and I is an ideal of A, it is known that $\overline{I} = \overline{IB} \cap A$ ([10, Proposition 1.6.1]). In particular, every element of $IB \cap A$ belongs to the integral closure of I. In what follows we show that a similar result holds for the weak subintegral closure of an ideal. We first note the persistence property of the weak subintegral closure.

Proposition 4.1. *If $f : A \to B$ is a ring homomorphism, I is an ideal of A and $a \in {}^*I$, then $f(a) \in {}^*(IB)$.*

Proof. Let $\overline{A[It]}$ and $\overline{B[(IB)t]}$ denote the integral closures of $A[It]$ and $B[(IB)t]$ in $A[t]$ and $B[t]$, respectively. The map f induces the natural map

$$\overline{A[It]} \otimes_{A[It]} \overline{A[It]} \xrightarrow{\tilde{f}} \overline{B[(IB)t]} \otimes_{B[(IB)t]} \overline{B[(IB)t]}.$$

Since $at \otimes 1 - 1 \otimes at$ is nilpotent in $\overline{A[It]} \otimes_{A[It]} \overline{A[It]}$, it follows that $\tilde{f}(at \otimes 1 - 1 \otimes at) = [f(a)t] \otimes 1 - 1 \otimes [f(a)t]$ is nilpotent as well, and therefore $f(a) \in {}^*(IB)$. $\square$

Theorem 4.2. *Let $A \subseteq B$ be a weakly subintegral extension of rings and I an ideal of A. Then ${}^*(IB) \cap A = {}^*I$.*

Proof. With the same notation used above, consider the natural maps

$$\overline{A[It]} \otimes_{A[It]} \overline{A[It]} \xrightarrow{\phi} \overline{B[(IB)t]} \otimes_{A[It]} \overline{B[(IB)t]} \xrightarrow{\theta} \overline{B[(IB)t]} \otimes_{B[(IB)t]} \overline{B[(IB)t]}.$$

Since $A \subseteq B$ is integral, the extension $\overline{A[It]} \subseteq \overline{B[(IB)t]}$ is integral, too. By (2.3) it follows that the map ϕ has nilpotent kernel. We next show that $\operatorname{Ker}\theta$ is nilpotent as well. By (2.4), $\operatorname{Ker}\theta$ is generated by $\{c \otimes 1 - 1 \otimes c \mid c \in B[(IB)t]\}$, so it is enough to show that every element of the form $\beta t^n \otimes 1 - 1 \otimes \beta t^n$ with $\beta \in I^n B$ is nilpotent in $\overline{B[(IB)t]} \otimes_{A[It]} \overline{B[(IB)t]}$. We show this by induction on n. For $n = 0$, since $\beta \in B$ and the extension $A \subseteq B$ is weakly subintegral, the element $\beta \otimes 1 - 1 \otimes \beta$ is nilpotent in $B \otimes_A B$, hence in $\overline{B[(IB)t]} \otimes_{A[It]} \overline{B[(IB)t]}$. For $n = 1$, let $\beta \in IB$. After choosing a set of generators for the ideal I, say $I = (a_1, \ldots, a_k)$, write $\beta = a_1 b_1 + \cdots + a_k b_k$ for some $b_1, \ldots, b_k \in B$. Then, in $\overline{B[(IB)t]} \otimes_{A[It]} \overline{B[(IB)t]}$, we have

$$\begin{aligned} \beta t \otimes 1 - 1 \otimes \beta t &= \sum_{i=1}^{k} [(a_i b_i t) \otimes 1 - 1 \otimes (a_i b_i t)] = \sum_{i=1}^{k} [(a_i b_i t) \otimes 1 - (a_i t) \otimes b_i] \\ &= \sum_{i=1}^{k} ((a_i t) \otimes 1)(b_i \otimes 1 - 1 \otimes b_i). \end{aligned}$$

However, since the extension $A \subseteq B$ is weakly subintegral, every element $b_i \otimes 1 - 1 \otimes b_i$ is nilpotent in $B \otimes_A B$, and therefore $\beta t \otimes 1 - 1 \otimes \beta t$ is nilpotent in $\overline{B[(IB)t]} \otimes_{A[It]} \overline{B[(IB)t]}$.

For $n \geq 2$, it is enough to prove that $(\beta_1 \cdots \beta_n)t^n \otimes 1 - 1 \otimes (\beta_1 \cdots \beta_n)t^n$ is nilpotent in $\overline{B[(IB)t]} \otimes_{A[It]} \overline{B[(IB)t]}$ for every $\beta_1, \ldots, \beta_n \in IB$. Since

$$(\beta_1 \cdots \beta_n)t^n \otimes 1 - 1 \otimes (\beta_1 \cdots \beta_n)t^n = [(\beta_1 \cdots \beta_{n-1})t^{n-1} \otimes 1 - 1 \otimes (\beta_1 \cdots \beta_{n-1})t^{n-1}](\beta_n t \otimes 1) + (\beta_n t \otimes 1 - 1 \otimes \beta_n t)[1 \otimes (\beta_1 \cdots \beta_{n-1})t^{n-1}],$$

this follows from the induction hypothesis.

Now let $\alpha \in {}^*(IB) \cap A$. Note that $\alpha \in \overline{IB} \cap A$, hence $\alpha \in \overline{I}$. The element $\alpha t \otimes 1 - 1 \otimes \alpha t$ is nilpotent in $\overline{B[(IB)t]} \otimes_{B[(IB)t]} \overline{B[(IB)t]}$ and since the map $\theta \circ \phi$ has nilpotent kernel, it follows that $\alpha t \otimes 1 - 1 \otimes \alpha t$ is nilpotent in $\overline{A[It]} \otimes_{A[It]} \overline{A[It]}$, showing that $\alpha \in {}^*I$. The inclusion ${}^*I \subseteq {}^*(IB) \cap A$ follows from Proposition 4.1. □

Proposition 4.3. *Let $A \subseteq B$ be an extension of rings and let $z \in A$ such that z is not a zero-divisor of B. Then*

$${}^*_{B_z}(A_z) \cap B = ({}^*_B A :_B z).$$

Proof. Let $\alpha \in {}^*_{B_z}(A_z) \cap B$, so $\alpha \otimes 1 - 1 \otimes \alpha$ is nilpotent in $B_z \otimes_{A_z} B_z \cong (B \otimes_A B)_z$. Then there exists n such that $z^n(\alpha \otimes 1 - 1 \otimes \alpha)$ is nilpotent in $B \otimes_A B$, which implies that $z(\alpha \otimes 1 - 1 \otimes \alpha)$ is nilpotent in $B \otimes_A B$. This shows that $\alpha z \in {}^*_B A$. The other inclusion also follows immediately. If $\alpha \in ({}^*_B A :_B z)$, then $z\alpha \otimes 1 - 1 \otimes z\alpha = z(\alpha \otimes 1 - 1 \otimes \alpha)$ is nilpotent in $B \otimes_A B$, which implies that $\alpha \otimes 1 - 1 \otimes \alpha$ is nilpotent in $B_z \otimes_{A_z} B_z$. □

Corollary 4.4. *Let I be an ideal of A and $z \in A$ a regular element. Then*

$$({}^*I)_z \cap \overline{I} = ({}^*I : z) \cap \overline{I}.$$

Proof. We apply the previous proposition for the ring extension $A[It] \subseteq \overline{A[It]}$, where $\overline{A[It]}$ is the integral closure of $A[It]$ in $A[t]$. By considering the equality obtained for the homogeneous components in t-degree one, the conclusion follows. □

Proposition 4.5. *Let I, J be ideals of a noetherian ring A such that J contains a regular element. Then*

$$({}^*I : J^\infty) \cap \overline{I} = ({}^*I : J) \cap \overline{I}.$$

Proof. If J is principal, say $J = (z)$ where z is a regular element, this follows immediately from Corollary 4.4, as $({}^*I)_z \cap A = ({}^*I : z^\infty)$. For the general case, since J contains regular elements, from the implication $(1 \Rightarrow 4)$ of [9, Theorem 7.2] it follows that there exist $z_1, \ldots, z_n$ regular elements with $J = (z_1, \ldots, z_n)$. Then $({}^*I : J^\infty) \cap \overline{I} = \bigcap_{i=1}^n ({}^*I : z_i^\infty) \cap \overline{I} = \bigcap_{i=1}^n ({}^*I : z_i) \cap \overline{I} = ({}^*I : J) \cap \overline{I}$. □

Corollary 4.6. *Let $(A, \mathfrak{m})$ be a local ring of positive depth and I an ideal of A. If the length $\lambda(\overline{I}/{}^*I)$ is finite, then $\mathfrak{m}\overline{I} \subseteq {}^*I$.*

Proof. If $\lambda(\overline{I}/{}^*I) < \infty$, then $\overline{I} \subseteq ({}^*I : \mathfrak{m}^\infty)$. By Proposition 4.5 we then obtain $\overline{I} \subseteq ({}^*I : \mathfrak{m})$. □

Remark 4.7. In the case when I is $\mathfrak{m}$-primary, the inclusion $\mathfrak{m}\overline{I} \subseteq {}^*I$ also follows from [5, Proposition 5.3], as $\mathfrak{m}\overline{I} \subseteq \overline{\mathfrak{m}I} \subseteq I^{-\mathrm{sp}} = I_> \subseteq {}^*I$.

We also have the following.

Corollary 4.8. *For every regular ideal I of a noetherian ring A, the ideal $({}^*I : \overline{I})$ is radical. In particular,*

$$\overline{I}\sqrt{I : \overline{I}} \subseteq {}^*I.$$

Proof. Let $J = \sqrt{{}^*I : \overline{I}}$. Note that $I \subseteq J$, so J contains a regular element. Since $\overline{I} \subseteq ({}^*I : J^\infty)$, by Proposition 4.5 we have $\overline{I} \subseteq ({}^*I : J)$, hence $J \subseteq ({}^*I : \overline{I})$. □

Remark 4.9. As we already mentioned, Gaffney and Vitulli [7, Proposition 4.4] proved that $I_> \subseteq {}^*I$ by using the equational description of ${}^*I$. Equivalently, if $a^n \in I^{n+1}$ for $n \gg 0$, then $a \in {}^*I$. In fact, a careful examination of their argument shows that if $a^n \in I^n$ for $n \gg 0$, then $a \in {}^*I$. We note here that the same result can also be recovered from the tensor product characterization of ${}^*I$. Indeed, assume that there exists n_0 such that $a^n \in I^n$ for $n \geq n_0$. We claim that $(at \otimes 1 - 1 \otimes at)^{2n_0 - 1} = 0$ in $\overline{A[It]} \otimes_{A[It]} \overline{A[It]}$, and hence $a \in {}^*I$. First note that from $a^n \in I^n$ for $n \geq n_0$ we have $a \in \overline{I}$. For $0 \leq k \leq n_0 - 1$ we have

$(at)^{n_0+k}, (at)^{2n_0-1} \in A[It]$ and therefore

$$\begin{aligned}(at)^{n_0+k} \otimes (at)^{n_0-1-k} &= 1 \otimes [(at)^{n_0+k}(at)^{n_0-1-k}] \\ &= 1 \otimes (at)^{2n_0-1} = (at)^{2n_0-1} \otimes 1 \\ &= (at)^{n_0-1-k} \otimes (at)^{n_0+k}.\end{aligned}$$

From the binomial expansion we then obtain

$$(at\otimes 1 - 1\otimes at)^{2n_0-1} = \sum_{k=0}^{n_0-1} (-1)^{n_0-k} \binom{2n_0-1}{n_0+k} \left[(at)^{n_0+k} \otimes (at)^{n_0-k-1} - (at)^{n_0-k-1} \otimes (at)^{n_0+k} \right] = 0.$$

5. Weak normalizations of multi-Rees algebras

Our main goal in this section is to describe the weak normalization of a multi-Rees algebra $\mathcal{R}(I_1, \ldots, I_k) = A[I_1t_1, \ldots, I_kt_k]$ in $A[t_1, \ldots, t_k]$. For integral closure, it is well known that $\overline{\mathcal{R}(I_1, \ldots, I_k)} = \oplus_{m_1,\ldots,m_k \geq 0} \overline{I_1^{m_1} \cdots I_k^{m_k}} t_1^{m_1} \cdots t_k^{m_k}$. We obtain a similar characterization for the weak normalization.

Theorem 5.1. *Let $A \subseteq B$ be a ring extension. Let $I_1, \ldots, I_k$ be ideals of A with multi-Rees algebra $\mathcal{R} = \mathcal{R}(I_1, \ldots, I_k) = A[I_1t_1, \ldots, I_kt_k]$ Then*

$$^*_{B[t_1,\ldots,t_k]}\mathcal{R} = \bigoplus_{(m_1,\ldots,m_k)\in\mathbb{N}^k} {}^*_B(I_1^{m_1} \cdots I_k^{m_k}) t^{m_1} \cdots t_k^{m_k}.$$

Proof. Let $\Sigma = \mathbb{Z}^k$. For $i = (m_1, \ldots, m_k) \in \mathbb{N}^k$, we denote $t^i = t_1^{m_1} \cdots t_k^{m_k}$ and $I^i = I_1^{m_1} \cdots I_k^{m_k}$. Also denote by $\mathcal{S}$ the integral closure of $\mathcal{R}$ in $B[t_1, \ldots, t_k]$. Since ${}^*_{B[t_1,\ldots,t_k]}\mathcal{R}$ is a Σ-graded subring of $\mathcal{S} \subseteq B[t_1, \ldots, t_k]$, we can write ${}^*_{B[t_1,\ldots,t_k]}\mathcal{R} = \bigoplus_{i\in\Sigma} L_i t^i$. Fix $\varepsilon = (m_1, \ldots, m_k) \in \mathbb{N}^k$. For $\Delta = \mathbb{Z}\varepsilon$, by Theorem 2.14, we have $L_\varepsilon = [{}^*_{(\mathcal{S}^\Delta)}(\mathcal{R}^\Delta)]_\varepsilon$. On the other hand, $\mathcal{R}^\Delta = \bigoplus_{n\in\mathbb{N}} I^{n\varepsilon} t^{n\varepsilon}$ is the classic (single) Rees algebra of the ideal I^ε, and $\mathcal{S}^\Delta$ is its integral closure in $B[t^\varepsilon]$. If we consider these algebras with their natural $\mathbb{N}$-grading, i.e., $[\mathcal{R}^\Delta]_n = I^{n\varepsilon} t^{n\varepsilon}$, then L_ε is obtained from degree one component of ${}^*_{(\mathcal{S}^\Delta)}(\mathcal{R}^\Delta)$, and by Theorem 2.18 and Remark 2.20 it follows that $L_\varepsilon = {}^*_B(I^\varepsilon) = {}^*_B(I_1^{m_1} \cdots I_k^{m_k})$, finishing the proof. □

The following immediate consequence recovers a result of Vitulli and Leahy [22, Proposition 2.11], who obtained it by using very technical characterizations of ${}^*_B I$ (see [22, Theorem 2.10]).

Corollary 5.2. *Let $A \subseteq B$ be a ring extension and let I, J be ideals of A. Then*

$$({}^*_B I)({}^*_B J) \subseteq {}^*_B (IJ),$$

and therefore

$${}^*_B \big(({}^*_B I)({}^*_B J)\big) = {}^*_B (IJ).$$

Proof. From the graded structure of ${}^*_{B[t_1,t_2]} A[It_1, Jt_2]$ proved in Theorem 5.1 it follows that $({}^*_B It_1)({}^*_B Jt_2) \subseteq {}^*_B (IJ)t_1t_2$. □

6. The weak normalization of the extended Rees algebra

It is well known that the integral closures of the Rees and extended Rees algebras have identical homogeneous components of non-negative degree. A similar property holds for their weak normalizations.

Proposition 6.1. *Let $A \subseteq B$ be a ring extension and I an ideal of A. The weak normalizations of $A[It]$ and $A[It, t^{-1}]$ in $B[t]$ and $B[t, t^{-1}]$, respectively, have the same homogeneous components of degree $n > 0$.*

Proof. Let $C := {}^*_B A$. From part (a) of Corollary 2.21 we have ${}^*_{B[t]}(A[It]) = {}^*_{C[t]}(C[(IC)t]);$ similarly, one can also show that ${}^*_{B[t,t^{-1}]}(A[It, t^{-1}]) = {}^*_{C[t,t^{-1}]}(C[(IC)t, t^{-1}])$. Therefore, after replacing C with A, it is enough to prove that $[{}^*_{A[t]}(A[It])]_n = [{}^*_{A[t,t^{-1}]}(A[It, t^{-1}])]_n$ for $n \geq 1$.

The inclusion "$\subseteq$" follows right away. Indeed, if $\alpha t^n \otimes 1 - 1 \otimes \alpha t^n$ is nilpotent in $\overline{A[It]} \otimes_{A[It]} \overline{A[It]}$, it is also nilpotent in $\overline{A[It, t^{-1}]} \otimes_{A[It,t^{-1}]} \overline{A[It, t^{-1}]}$.

Assume now that $\alpha t^n \in [{}^*_{A[t,t^{-1}]}(A[It, t^{-1}])]_n$, i.e., $(\alpha t^n \otimes 1 - 1 \otimes \alpha t^n)^{k_0} = 0$ in $\overline{A[It, t^{-1}]} \otimes_{A[It,t^{-1}]} \overline{A[It, t^{-1}]}$ for some $k_0 \geq 1$. Then $(\alpha t^n \otimes 1 - 1 \otimes \alpha t^n)^{k_0} = 0$ in $S' \otimes_{A[It,t^{-1}]} S'$ for some finitely generated $A[It, t^{-1}]$-submodule of $\overline{A[It, t^{-1}]}$, so there exists $u \geq 1$ such that $(\alpha t^n \otimes 1 - 1 \otimes \alpha t^n)^{k_0} = 0$ in $A[\overline{I}t, \ldots, \overline{I^u}t^u, t^{-1}] \otimes_{A[It,t^{-1}]} A[\overline{I}t, \ldots, \overline{I^u}t^u, t^{-1}]$. Let $T := A[\overline{I}t, \ldots, \overline{I^u}t^u]$ and

$T' := A[\overline{I}t, \ldots, \overline{I^u}t^u, t^{-1}]$ and consider the canonical graded homomorphism

$$T \otimes_{A[It]} T \xrightarrow{\eta} T' \otimes_{A[It,t^{-1}]} T'.$$

Let $\{m_i\}_{1\leq i\leq r}$ be a finite set of non-zero homogeneous generators of T as an $A[It]$-module and let $s = \max\{\deg m_i \mid 1 \leq i \leq r\}$. Note that $\{m_i\}_{1\leq i\leq r}$ also generate T' as an $A[It, t^{-1}]$-module. We claim that for $k \geq 2s$ every homogeneous element from $[\operatorname{Ker}\eta]_k$ is zero.

Let $y = \sum_{i=1}^r m_i \otimes x_i \in [\operatorname{Ker}\eta]_k$ with $x_i \in T_{k-\deg m_i}$ homogeneous elements. By Lemma 2.7, there exist homogeneous elements $a_{ij} \in [A[It, t^{-1}]]_{k-\deg m_i-\deg m_j}$ $(1 \leq i, j \leq r)$ such that $x_i = \sum_{j=1}^r a_{ij} m_j$ for every i and $\sum_{i=1}^r m_i a_{ij} = 0$ for every j. However, since $k \geq 2s$, we have $k - \deg m_i - \deg m_j \geq 0$ for all i, j, so $a_{ij} \in A[It]$. By Lemma 2.7 again, this implies that $y = \sum_{i=1}^r m_i \otimes x_i = 0$ in $T \otimes_{A[It]} T$.

Now let $k \geq \max\{k_0, 2s\}$. We have $(\alpha t^n \otimes 1 - 1 \otimes \alpha t^n)^k = 0$ in $T' \otimes_{A[It,t^{-1}]} T'$, so $(\alpha t^n \otimes 1 - 1 \otimes \alpha t^n)^k \in \operatorname{Ker}\eta$. Since $nk \geq 2s$, from what we just proved above it follows that $(\alpha t^n \otimes 1 - 1 \otimes \alpha t^n)^k = 0$ in $T \otimes_{A[It]} T$. Then $(\alpha t^n \otimes 1 - 1 \otimes \alpha t^n)^k = 0$ in $\overline{A[It]} \otimes_{A[It]} \overline{A[It]}$, too, showing that $\alpha t^n \in [{}^*_{A[t]}(A[It])]_n$. $\square$

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Department of Mathematics 2750, North Dakota State University, PO BOX 6050, Fargo, ND 58108-6050, USA

Email address: catalin.ciuperca@ndsu.edu