ASYMPTOTIC PRIMES OF $S_2$-FILTRATIONS

CĂTĂLIN CIUPERCĂ

Abstract. Let $A$ be a noetherian ring with total ring of fractions $Q(A)$ and $S = \oplus_{n \in \mathbb{Z}} I_n t^n$ a noetherian graded ring such that $A[t^{-1}] \subseteq S \subseteq Q(A)[t, t^{-1}]$ and $S$ satisfies the $(S_2)$ property of Serre. Under mild conditions on the ring $A$, we study the behavior of the sets of associated prime ideals Ass$(A/I_n \cap A)$ for $n \geq 1$. In particular, we consider the case when $S$ is the $S_2$-ification of the extended Rees algebra of an ideal $I$. As applications, we obtain several results regarding the asymptotic behavior of Ass$(A/I^n)$ for certain ideals of analytic deviation one. We also prove several consequences about the symbolic powers of a prime ideal.

1. Introduction

Let $A$ be a noetherian ring and $I$ an ideal in $A$. Answering a question of Ratliff regarding the behavior of Ass$(A/I^n)$ for $n$ large, Brodmann proved that the sequence of sets Ass$(A/I^n)$ stabilizes for $n$ large enough. In the same vein, if we consider the filtration of integral closures $\{\overline{I^n}\}_{n \geq 1}$, results of Ratliff show that the sets Ass$(A/\overline{I^n})$ stabilize as well. Moreover, Ass$(A/\overline{I^n}) \subseteq$ Ass$(A/\overline{I^{n+1}})$ for all $n$, a property which is not always satisfied by the sets Ass$(A/I^n)$. If $A^*(I)$ and $\overline{A}^*(I)$ denote the stabilizing sets of Ass$(A/I^n)$ and Ass$(A/\overline{I^n})$, respectively, it is also known that $\overline{A}^*(I) \subseteq A^*(I)$. We refer to the monograph of McAdam [15] for a detailed exposition of these properties.

The asymptotic behavior of these sets of associated primes is best studied by considering the extended Rees algebra $R = A[It, t^{-1}]$. If $\overline{R}$ denotes the integral closure of $R$ in its total quotient ring, we have $\overline{R} \subseteq Q(A)[t, t^{-1}]$, where $Q(A)$ is the total ring of fractions of $A$, and $\overline{R} \cap A[t, t^{-1}] = \oplus_{n \in \mathbb{Z}} \overline{I^n} t^n$. The properties $(R_1)$ and $(S_2)$ of Serre derived from the integral closedness of the ring $\overline{R}$ can then be exploited to deduce the nice behavior of the sets Ass$(A/\overline{I^n})$. In this paper we consider finite graded extensions $S = \oplus_{n \in \mathbb{Z}} I_n t^n$ of the extended

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Rees algebra \( A[It, t^{-1}] \) inside \( Q(A)[t, t^{-1}] \) that only satisfy the \((S_2)\) property of Serre. Under mild conditions on the ring \( A \), we show that the filtration of ideals \( \{I_n \cap A\}_{n \geq 1} \), which we refer to as an \( S_2 \)-filtration, has similar asymptotic properties with respect to its associated prime ideals. More precisely, the sets \( \text{Ass}(A/I_n \cap A) \) form an increasing sequence that eventually stabilizes and \( \bigcup_{n \geq 1} \text{Ass}(A/I_n \cap A) = \bigcup_{n \geq 1} \text{Ass}(A/I^n) \) (Corollary 2.8). It is not the purpose of this note to recover the already known results about the asymptotic behavior of \( \text{Ass}(A/I^n) \).

In fact, many of our arguments assume that the ring \( A \) is a universally catenary domain, a constraint not imposed in the work of Ratliff regarding the behavior of \( \text{Ass}(A/I^n) \). Our main goal is to apply these properties in a minimal finite birational extension of \( R \) that satisfies the \((S_2)\) property, the so-called \( S_2 \)-ification of \( R \) (Corollary 2.15). With respect to the arguments used in our proofs, since we are only assuming the \((S_2)\) property, we can no longer employ the use of certain discrete valuation rings which, in the case of the filtration \( \{I_n\}_{n \geq 1} \), were obtained by localizing \( R \) at the minimal prime ideals of \( t^{-1}R \) (cf. \([15, 3.1–3.3]\)). Additionally, some other mild assumptions on the ring need to be made in order to ensure the existence of an \( S_2 \)-ification.

Another motivating result for our study is a characterization due to Ratliff \([17, \text{Theorem 4.3}]\) of the Cohen-Macaulay rings in terms of the sets \( A^*(I) \) and \( \overline{A}^*(I) \). More precisely, if \( A \) is a locally formally equidimensional ring, then \( A \) is Cohen-Macaulay if and only if \( A^*(I) = \overline{A}^*(I) \) for all ideals \( I \) of the principal class. By using our results regarding \( A^*(I) \) when \( R \) satisfies \((S_2)\), we conclude that \( A \) is Cohen-Macaulay if and only if \( A[It, t^{-1}] \) satisfies \((S_2)\) for every ideal \( I \) of the principal class (Proposition 2.17).

All these results are obtained in Section 2 where we develop the main ideas in the general context of graded noetherian algebras \( S = \bigoplus_{n \in \mathbb{Z}} I_n t^n \) such that \( A[t^{-1}] \subseteq S \subseteq Q(A)[t, t^{-1}] \) and \( S \) satisfies the \((S_2)\) property. In Section 3 we obtain several consequences regarding the symbolic powers of a prime ideal. In particular, if \( p \) is a prime ideal in a universally catenary domain \( A \) such that the associated graded ring \( G_{pA_p}(A_p) \) satisfies \((S_1)\), we show that \( \overline{A}^*(p) = \{p\} \) if and only if the \( n \)-th symbolic power \( p^{(n)} \) coincides with the degree \( n \) component in \( A \) of the \( S_2 \)-ification of \( R \) for all \( n \geq 1 \) (Theorem 3.4). This improves a similar characterization of Huckaba who, under the global assumption that \( G_p(A) \) satisfies \((S_1)\), proved that \( \overline{A}^*(p) = \{p\} \) if and only if \( p^{(n)} = p^n \) for all \( n \) (Corollary 3.5). We note that
the equimultiple ideals are important cases of ideals that satisfy the condition \( A^*(p) = \{p\} \).

We also provide an example of an equimultiple ideal \( p \) in a Cohen-Macaulay domain such that \( G_{p,A_p}(A_p) \) satisfies \((S_1)\) but \( G_p(A) \) does not, and compute several symbolic powers of \( p \) by using the \( S_2 \)-ification of its extended Rees algebra. In Section 4 we recover a result of Brodmann showing that, for an almost complete intersection ideal \( I \), the sets \( \text{Ass}(A/I^n) \) form an increasing sequence and the height of any prime in \( A^*(I) \) is at most \( \text{ht} I + 1 \) (Proposition 4.3). Similar properties are also obtained for certain ideals \( I \) of analytic spread \( \ell(I) = \text{ht} I + 1 \) (Proposition 4.4). In both situations, the main observation is that the extended Rees algebras of the ideals involved satisfy the \((S_2)\) property, which allows us to apply Corollary 2.10.

2. \( S_2 \)-filtrations

Throughout this paper all the rings are commutative with identity. A noetherian local ring \((A, m)\) is said to be formally equidimensional if all the minimal primes of the completion \( \hat{A} \) have the same dimension. A noetherian ring is called locally formally equidimensional if all of its localizations are formally equidimensional. It is known that a locally formally equidimensional ring is universally catenary and, if \( A \) is a noetherian domain, \( A \) is locally formally equidimensional if and only if \( A \) is universally catenary. We refer to the Appendix of [12] for a brief account of these properties and for terminology and concepts not otherwise explained in this paper.

2.1. The \((S_i)\) property. Let \( M \) be a finitely generated module over a noetherian ring \( A \). We say that \( M \) satisfies Serre’s \((S_i)\) property if for every \( p \in \text{Spec}(A) \) we have

\[
\text{depth } M_p \geq \min \{i, \dim M_p\}.
\]

We caution that a slightly different definition is sometimes used in the literature by requiring the stronger condition \( \text{depth } M_p \geq \min \{i, \text{ht } p\} \). The conditions are clearly equivalent if \( \text{Ann}_A M = (0) \).

We say that the ring \( A \) satisfies the \((S_2)\) property if \( A \) satisfies the \((S_2)\) property as an \( A \)-module. Equivalently, \( A \) has no embedded prime ideals (i.e. \( A \) satisfies the \((S_1)\) property) and, for every regular element \( x \in A \), the ring \( A/xA \) has no embedded prime ideals.
2.2. If $A \hookrightarrow B$ is a finite extension of noetherian rings and $B$ satisfies the $(S_2)$ property as an $A$-module, then $B$ satisfies the $(S_2)$ property as a ring.

Moreover, if the extension satisfies the condition

$$\text{ht } q_1 = \text{ht } q_2 \text{ for every } q_1, q_2 \in \text{Spec } B \text{ with } q_1 \cap A = q_2 \cap A,$$

then the converse also holds, i.e., $B$ satisfies $(S_2)$ as a ring if and only if $B$ satisfies $(S_2)$ as an $A$-module ([14, 5.7.11]). In particular, if $A$ is a universally catenary domain and $A \hookrightarrow B$ is a finite birational extension, since $\text{ht } q = \text{ht}(q \cap A)$ for every $q \in \text{Spec } B$ ([12, 4.8.6], it follows that $B$ satisfies $(S_2)$ as a ring if and only if $B$ satisfies $(S_2)$ as an $A$-module.

In the above context, when we say that $B$ satisfies the $(S_2)$ property we mean that $B$ satisfies the $(S_2)$ property as a ring.

**Definition 2.3.** Let $A$ be a noetherian ring with total ring of fractions $Q(A)$. If $S = \oplus_{n \in \mathbb{Z}} I_n t^n$ is a noetherian graded ring with $A[t^{-1}] \subseteq S \subseteq Q(A)[t, t^{-1}]$ and $S$ satisfies the $(S_2)$ property, we say that the family of ideals $F = \{I_n \cap A\}_{n \geq 1}$ is an $S_2$-filtration of ideals in $A$.

The following Proposition contains results that are well-known in the case when $S$ is the integral closure of an extended Rees algebra $A[It, t^{-1}]$. Our statements are adaptations for the more general context of certain graded algebras that only satisfy the $(S_2)$ property. Most arguments used in its proof are also variations of known techniques.

**Proposition 2.4.** Let $A$ be a noetherian ring and $S = \oplus_{n \in \mathbb{Z}} I_n t^n$ a noetherian graded ring such that $A[t^{-1}] \subseteq S \subseteq Q(A)[t, t^{-1}]$ and $S$ satisfies the $(S_2)$ property. Let $Q_1, Q_2, \ldots, Q_m$ be the minimal prime ideals of the $S$-module $S/t^{-1}S$. The following are true:

(a) $I_n \cap A = \bigcap_{i=1}^m t^{-n} S_Q \cap A$ for all $n \geq 1$;
(b) $\bigcup_{n \geq 1} \text{Ass}(A/I_n \cap A) \subseteq \{Q_1 \cap A, \ldots, Q_m \cap A\}$;
(c) If $S \subseteq A[t, t^{-1}]$, then

$$\bigcup_{n \geq 1} \text{Ass}(A/I_n) = \{Q_1 \cap A, \ldots, Q_m \cap A\};$$
(d) If \( A \) is a universally catenary domain, \( S \) is a finitely generated \( A \)-algebra, and \( S \) is finite over \( S \cap A[t, t^{-1}] \), then
\[
\bigcup_{n \geq 1} \text{Ass}(A/I_n \cap A) = \{Q_1 \cap A, \ldots, Q_m \cap A\}.
\]

Proof. (a) Since \( S \) satisfies the \((S_2)\) property, for each \( n \geq 1 \) we have \( \text{Ass}(S/t^{-n}S) = \text{Min}(S/t^{-1}S) = \{Q_1, \ldots, Q_m\} \), so \( t^{-n}S = \bigcap_{i=1}^{m} t^{-n}S_{Q_i} \cap S \). As \( I_n = t^{-n}S \cap Q(A) \), the conclusion follows.

(b) The ideal \( t^{-n}S_{Q_i} \) is a \( Q_iS_{Q_i} \)-primary ideal in \( S_{Q_i} \), so \( t^{-n}S_{Q_i} \cap A \) is a primary ideal in \( A \). Then, for each \( n \),
\[
I_n \cap A = \bigcap_{i=1}^{m} (t^{-n}S_{Q_i} \cap A)
\]
is a (possibly redundant) primary decomposition of \( I_n \cap A \), and therefore
\[
\bigcup_{n \geq 1} \text{Ass}(A/I_n \cap A) \subseteq \{Q_1S_{Q_1} \cap A, \ldots, Q_mS_{Q_m} \cap A\} = \{Q_1 \cap A, \ldots, Q_m \cap A\}.
\]

(c) Let \( Q \) be a minimal prime ideal of \( S/t^{-1}S \). Write \( Q = (t^{-1}S :_S at^s) \) for some homogeneous element \( at^s \in S (a \in I_s) \). Then
\[
Q \cap A = (t^{-1}S :_A at^s) = (I_{s+1} :_A a),
\]
which shows that \( Q \cap A \in \text{Ass}(A/I_{s+1}) \).

(d) Let \( Q \) be a minimal prime ideal of \( S/t^{-1}S \). Denote \( S' = S \cap A[t, t^{-1}] \) and \( Q' = Q \cap S' \).
Since \( A \subseteq S' \subseteq S \), \( S \) is a finitely generated \( A \)-algebra, and \( S \) is a module-finite extension of \( S' \), by a result of Artin and Tate [1] Theorem 1], the ring \( S' \) is a finitely generated \( A \)-algebra, and therefore a universally catenary domain. Then, by [12] Proposition 4.8.6], we have \( \text{ht} Q' = \text{ht} Q = 1 \), so \( Q' \) is a minimal prime of \( S'/t^{-1}S' \). As in part (c), we can now write \( Q' = (t^{-1}S' :_{S'} at^s) \) for some homogeneous element \( at^s \in S' (a \in I_s \cap A) \). Then
\[
Q \cap A = Q' \cap A = (t^{-1}S' :_{A} at^s) = ((I_{s+1} \cap A) :_A a),
\]
which shows that \( Q \cap A \in \text{Ass}(A/I_{s+1} \cap A) \). \( \square \)
Remark 2.5. A similar argument to the one used above in part (c) shows that for every ideal $I$ in a noetherian ring $A$ we have

$$\bigcup_{n \geq 1} \text{Ass}(A/I^n) \supseteq \{P_1 \cap A, \ldots, P_r \cap A\},$$

where $\{P_1, \ldots, P_r\} = \text{Min}(A[It, t^{-1}]/(t^{-1}))$.

In order to prove the main results of this section we need the following lemma.

**Lemma 2.6.** Let $(A, \mathfrak{m})$ be a formally equidimensional local ring and let $S = \oplus_{n \in \mathbb{Z}} I_n t^n$ be a noetherian graded ring such that $A[t^{-1}] \subseteq S \subseteq Q(A)[t, t^{-1}]$ and $\text{ht}(I_1 \cap A) \geq 1$. Let $R = A[(I_1 \cap A)t, t^{-1}] \subseteq S$. Assume that $S$ is integral over $R$ and that either

(i) $A$ is a domain; or

(ii) $S \subseteq A[t, t^{-1}]$.

Then

$$\text{ht}((I_1 \cap A)t, t^{-1})S \geq 2.$$  

**Proof.** By contradiction, assume that there exists a prime $Q$ containing $((I_1 \cap A)t, t^{-1})S$ such that $\text{ht} Q = 1$. Let $P = Q \cap R$.

If the condition (i) is satisfied, then $R$ is a universally catenary domain and $R \hookrightarrow S$ is an integral extension. By using again [12, 4.8.6], we obtain $\text{ht} P = \text{ht} Q = 1$, so $P$ is a prime in the extended Rees algebra $R$ that is minimal over $t^{-1}R$. As $A$ is formally equidimensional, by [12, 5.4.8] we have $\dim R/P = \dim A$. On the other hand, since $R/((I_1 \cap A)t, t^{-1})R \cong A/I_1 \cap A$, we have $\dim R/P \leq \dim A/I_1 \cap A$. This implies $I_1 \cap A = (0)$, contradiction.

We now assume that the condition (ii) is satisfied. We first show that every minimal prime ideal of $S$ is of the form $pA[t, t^{-1}] \cap S$ with $p$ a minimal prime of $A$. To see this, for every ideal $J$ in $A$, denote $J_S = JA[t, t^{-1}] \cap S$. One can immediately check that if $p$ is a prime ideal in $A$, then $p_S$ is a prime ideal in $S$, and if $q$ is a $p$-primary ideal in $A$, then $q_S$ is a $p_S$-primary ideal in $S$. Moreover, $J_S \cap A = J$ for every ideal $J$ in $A$. Consequently, if $(0) = q_1 \cap \ldots \cap q_r$ is an irredundant primary decomposition of the zero ideal in $A$, then $(0) = (q_1)_S \cap \ldots \cap (q_r)_S$ is an irredundant primary decomposition of the zero ideal in $S$. This
shows that the ideals \( \{ p_S \mid p \in \text{Min}(A) \} \) are all the minimal prime ideals of \( S \). Similarly, the ideals \( \{ p_R \mid p \in \text{Min}(A) \} \) are all the minimal prime ideals of \( R \).

Now let \( p_S = pA[t, t^{-1}] \cap S \) be a minimal prime ideal of \( S \) such that \( Q \supseteq p_S \). Then \( p_R = p_S \cap R \) is a minimal prime ideal of \( R \) that is contained in \( P \). Note that \( R/p_R = R/(pA[t, t^{-1}] \cap R) \simeq (A/p)[(I_1 \cap A + p) t, t^{-1}] \) is the extended Rees algebra of the ring \( A/p \) with respect to the image of the ideal \( I_1 \cap A \). In particular,

\[
\dim R/p_R = \dim A/p + 1 = \dim A + 1.
\]

As \( A/p \) is a formally equidimensional local domain and hence universally catenary, \( R/p_R \) is also universally catenary. Then, if \( \mathfrak{M} = ((I_1 \cap A)t, m, t^{-1})R \) denotes the maximal homogeneous ideal of \( R \), we have

\[
(2.6.1) \quad \dim A + 1 = \dim R/p_R = \text{ht}(\mathfrak{M}/p_R) = \text{ht}(\mathfrak{M}/P) + \text{ht}(P/p_R) = \dim R/P + \text{ht}(P/p_R).
\]

Moreover, since \( R/p_R \hookrightarrow S/p_S \) is an integral extension, by [12, 4.8.6] we have \( \text{ht}(P/p_R) = \text{ht}(Q/p_S) = 1 \). Therefore \( (2.6.1) \) implies that \( \dim R/P = \dim A \). However, since \( R/((I_1 \cap A)t, t^{-1})R \cong A/I_1 \cap A \), we have \( \dim R/P \leq \dim A/I_1 \cap A \), contradicting \( \text{ht}(I_1 \cap A) \geq 1 \).

**Theorem 2.7.** Let \( A \) be a locally formally equidimensional ring and let \( S = \bigoplus_{n \in \mathbb{Z}} I_n t^n \) be a noetherian graded ring such that \( A[t^{-1}] \subseteq S \subseteq Q(A)[t, t^{-1}] \) and \( S \) satisfies the \( (S_2) \) property. Let \( R = A[(I_1 \cap A)t, t^{-1}] \subseteq S \). Assume that \( S \) is integral over \( R \) and that either

(i) \( A \) is a domain; or

(ii) \( S \subseteq A[t, t^{-1}] \) and \( \text{ht} I_1 \geq 1 \).

Then

\[
\text{Ass}(A/I_n \cap A) \subseteq \text{Ass}(A/I_{n+1} \cap A) \text{ for all } n \geq 1.
\]

**Proof.** First note that under the assumption (i) we may also assume that \( \text{ht}(I_1 \cap A) \geq 1 \), i.e. \( I_1 \cap A \neq (0) \), for otherwise we must have \( I_n = 0 \) for all \( n \geq 1 \).

Let \( Q \in \text{Ass}(A/I_n \cap A) \). After localizing at \( Q \) we may assume that \( A \) is a formally equidimensional local ring with maximal ideal \( Q \). Moreover, by passing to the faithfully flat extension \( A[X]_{QA[X]} \), we may also assume that the residue field of \( A \) is infinite. Write \( Q = \ldots\)
Then \( Q_{A/I} \) (a) Ass(\( S \)) Corollary 2.8. Let \( A \) be a locally formally equidimensional domain, \( I \) an ideal in \( A \), and \( S = \bigoplus_{n \in \mathbb{Z}} I_n t^n \) a noetherian graded ring such that \( A[It, t^{-1}] \subseteq S \subseteq Q(A)[t, t^{-1}] \) and \( S \) satisfies the \((S_2)\) property. Assume that \( S \) is a finite extension of \( A[It, t^{-1}] \). Then the following hold:

1. \( \text{Ass}(A/I_n \cap A) \subseteq \text{Ass}(A/I_{n+1} \cap A) \) for all \( n \geq 1 \);
2. \( \bigcup_{n \geq 1} \text{Ass}(A/I_n \cap A) = \{ Q \cap A \mid Q \in \text{Min}(S/t^{-1}S) \} \);
3. \( \bigcup_{n \geq 1} \text{Ass}(A/I_n \cap A) = \bigcup_{n \geq 1} \text{Ass}(A/T^n) \).

**Proof.** Parts (a) and (b) are immediate consequences of Theorem 2.7 and Proposition 2.4 (d).

For part (c), let \( T = \frac{A}{A[It, t^{-1}]} \) be the integral closure of \( A[It, t^{-1}] \) in its quotient field. Note that \( S \subseteq T = \bigoplus_{n \in \mathbb{Z}} \frac{I_n}{A} \bigcap \frac{At^n}{A} \subseteq Q(A)[t, t^{-1}] \) (\[12\], 5.2.4) and \( T \cap A[t, t^{-1}] = \bigoplus_{n \in \mathbb{Z}} T^n t^n \). From the classical result regarding the asymptotic primes of the filtration \( \{ T^n \}_{n \geq 1} \) we know that \( \bigcup_{n \geq 1} \text{Ass}(A/T^n) = \{ Q \cap A \mid Q \in \text{Min}(T/t^{-1}T) \} \) (see, for example, \[15\], Proposition 3.18). Because of (b), if we show that the contractions to \( S \) of all the primes in \( \text{Min}_T(T/t^{-1}T) \) give us all the primes in \( \text{Min}_S(S/t^{-1}S) \), the proof is finished. For this, note that \( S \) is a universally catenary domain and \( S \subseteq T \) is an integral extension. By \[12\], 4.8.6, for every prime \( Q \) minimal over \( t^{-1}T \) we have \( \text{ht}(Q \cap S) = \text{ht} Q = 1 \), so \( Q \cap S \) is minimal over \( t^{-1}S \). Moreover, if \( P \) is a prime minimal over \( t^{-1}S \), by the lying over property for integral extensions there exists a prime \( Q \) containing \( t^{-1}T \) such that \( P = Q \cap S \). We also have \( \text{ht} Q \leq \text{ht} P \), so \( Q \) must be minimal over \( t^{-1}T \).

When \( A \) is not necessarily a domain, but \( S \subseteq A[t, t^{-1}] \), we obtain similar conclusions in parts (a) and (b), and only one inclusion in part (c).

**Corollary 2.9.** Let \( A \) be a locally formally equidimensional ring, \( I \) an ideal in \( A \), and \( S = \bigoplus_{n \in \mathbb{Z}} I_n t^n \) a noetherian graded ring such that \( A[It, t^{-1}] \subseteq S \subseteq A[t, t^{-1}] \), \( \text{ht} I_1 \geq 1 \), and
$S$ satisfies the $(S_2)$ property. Assume that $S$ is a finite extension of $A[It,t^{-1}]$. Then the following hold:

(a) $\text{Ass}(A/I_n) \subseteq \text{Ass}(A/I_{n+1})$ for all $n \geq 1$;
(b) $\bigcup_{n \geq 1} \text{Ass}(A/I_n) = \{Q \cap A \mid Q \in \text{Min}(S/t^{-1}S)\}$;
(c) $\bigcup_{n \geq 1} \text{Ass}(A/I_n) \subseteq \bigcup_{n \geq 1} \text{Ass}(A/T^n)$.

Proof. Parts (a) and (b) follow from Theorem 2.7 (ii) and Proposition 2.4 (c). Moreover, by following the line of proof of part (c) of Corollary 2.8 without assuming that $A$ is domain, it is still true that every minimal prime of $S/t^{-1}S$ is contracted from a minimal prime of $T/t^{-1}T$, and therefore

$$\bigcup_{n \geq 1} \text{Ass}(A/I_n \cap A) \subseteq \bigcup_{n \geq 1} \text{Ass}(A/T^n).$$

□

In particular, when $S$ is an extended Rees algebra, we have the following.

**Corollary 2.10.** Let $A$ be a locally formally equidimensional ring and $I$ an ideal in $A$ with $\text{ht } I \geq 1$. Assume that the extended Rees algebra $A[It,t^{-1}]$ satisfies the $(S_2)$ property. The following hold:

(a) $\text{Ass}(A/I^n) \subseteq \text{Ass}(A/I^{n+1})$ for every $n \geq 1$;
(b) $\bigcup_{n \geq 1} \text{Ass}(A/I^n) = \{Q \cap A \mid Q \in \text{Min}(A[It,t^{-1}]/(t^{-1}))\}$;
(c) $\bigcup_{n \geq 1} \text{Ass}(A/I^n) = \bigcup_{n \geq 1} \text{Ass}(A/T^n)$.

Proof. Parts (a) and (b) are direct consequences of Corollary 2.9. From the classical result of Ratliff we have $\bigcup_{n \geq 1} \text{Ass}(A/T^n) = \overline{A^*}(I) \subseteq A^*(I)$ and, by part (a), $A^*(I) = \bigcup_{n \geq 1} \text{Ass}(A/I^n)$. This shows the inclusion “$\supseteq$” of part (c). The other inclusion follows from Corollary 2.9(c). □

**2.11. The $S_2$-ification of a noetherian domain.** For a noetherian domain $A$ with quotient field $Q(A)$, an $S_2$-ification of $A$ is a birational extension $S_2(A)$ of $A$ that is minimal among the finite birational extensions of $A$ that satisfy the $(S_2)$ property as $A$-modules. If $A$ has an $S_2$-ification, then it is unique. More precisely, $A$ has an $S_2$-ification $S_2(A)$ if and only if $C := \bigcap_{\text{ht } p = 1} A_p$ is a finite extension of $A$, in which case $C$ is the $S_2$-ification of $A$. 
If \( A \) is a universally catenary domain such that the extension \( A \subseteq \overline{A} \) is finite (which holds, for instance, when \( A \) is an analytically unramified local domain), then \( A \) has an \( S_2 \)-ification \([14, \text{5.11.2}]\). Also, if \( A \) has a canonical module \( \omega \), then \( \text{Hom}_A(\omega, \omega) \) is the \( S_2 \)-ification of \( A \) \([16, \text{Theorem 1.3}]\). We refer to \([6,16,19]\) for detailed accounts on \( S_2 \)-ification.

2.12. The \( S_2 \)-ification of \( A[I,t,t^{-1}] \). Let \( A \) be a noetherian domain and \( I \) an ideal in \( A \). Assume that \( R = A[I,t,t^{-1}] \) has an \( S_2 \)-ification \( S_2(R) \). (The \( S_2 \)-ification \( S_2(R) \) exists for a large class of rings; it does, for example, when \( A \) is a universally catenary analytically unramified local domain.) As explained in \([4, \text{2.3}]\), \( S_2(R) \) is a graded subring of \( Q(A)[t,t^{-1}] \), so that we can write \( S_2(R) = \oplus_{n \in \mathbb{Z}} I_n t^n \subseteq Q(A)[t,t^{-1}] \). The \( A \)-modules \( I_n \) are contained in the \( S_2 \)-ification \( S_2(A) \) of \( A \) \([4, \text{Lemma 2.4}]\). In particular, if \( A \) satisfies \((S_2)\), then \( I_n \) is an ideal in \( A \) for all \( n \geq 1 \).

**Definition 2.13.** For every ideal \( I \) in a ring \( A \) such that the conditions in \([2.12]\) are satisfied we define \( S_2(I) := I_1 \cap A \), the \( S_2 \)-closure of \( I \).

**Remark 2.14.** Since the \( n \)-th Veronese subring of \( S_2(A[I,t^{-1}]) \) is the \( S_2 \)-ification of the \( n \)-th Veronese subring of \( A[I,t^{-1}] \), we have \( S_2(I^n) = I_n \cap A \) for all \( n \geq 1 \).

From Corollary \([2.8]\) we have the following consequence regarding the asymptotic primes of the \( S_2 \)-filtration \( S_2(I^n) \).

**Corollary 2.15.** Let \( A \) be a universally catenary domain and \( I \) an ideal of \( A \). Assume that the extended Rees algebra \( R = A[I,t,t^{-1}] \) has an \( S_2 \)-ification . Then:

\( \text{(a)} \ \text{Ass}(A/S_2(I^n)) \subseteq \text{Ass}(A/S_2(I^{n+1})) \) for every \( n \geq 1 \);

\( \text{(b)} \ \bigcup_{n \geq 1} \text{Ass}(A/S_2(I^n)) = \bigcup_{n \geq 1} \text{Ass}(A/I^n) \).

**Remark 2.16.** Let \( A \) be a noetherian ring that satisfies the \((S_2)\) property and \( I \) an ideal. Consider the statements: (a) \( A[I,t^{-1}] \) satisfies the \((S_2)\) property; (b) \( G_I(A) \) satisfies the \((S_1)\) property ; and (c) \( A[I] \) satisfies the \((S_2)\) property. The following are true:

\( \text{(1)} \ (a) \iff (b) \), because \( G_I(A) \cong A[I,t,t^{-1}]/(t^{-1}) \);

\( \text{(2)} \ \text{If} \ ht I \geq 1 \text{, then} (c) \implies (b) \) \([3, \text{Theorem 1.5}]\).
(3) If \( I_p \) is principal for every \( p \in \text{Spec}(A) \) with \( \text{ht} \ p = 1 \), then (b) \( \implies \) (c) ([3, Theorem 1.5]); in particular, if \( \text{ht} \ I \geq 2 \), then (b) \( \iff \) (c).

The equivalence (a) \( \iff \) (e) in the following Proposition was originally established by Ratliff in [17]. In view of our study of the asymptotic primes of \( S_2 \)-filtrations, we are able to add several equivalent statements. In particular, the equivalence (a) \( \iff \) (c) gives a characterization of the Cohen-Macaulay rings in terms of the \( (S_2) \) property of their extended Rees algebras with respect to ideals of the principal class. Recall that an ideal \( I \) is said to be of the principal class if \( I \) can be generated by \( \text{ht} I \) elements.

**Proposition 2.17.** Let \( A \) be a locally formally equidimensional ring. The following are equivalent:

(a) \( A \) is Cohen-Macaulay;
(b) \( \mathcal{R} = A[It, t^{-1}] \) is Cohen-Macaulay for every ideal \( I \) of the principal class;
(c) \( \mathcal{R} = A[It, t^{-1}] \) satisfies the \( (S_2) \) property for every ideal \( I \) of the principal class;
(d) \( \bigcup_{n \geq 1} \text{Ass}(A/I^n) = \bigcup_{n \geq 1} \text{Ass}(A/T^n) \) for every ideal \( I \) of the principal class;
(e) \( \overline{A^*(I)} = \overline{A^*(I)} \) for every ideal \( I \) of the principal class.

**Proof.** The implication (a) \( \implies \) (b) holds because in a Cohen-Macaulay ring an ideal of the principal class is generated by a regular sequence. By Corollary 2.10, we also obtain (c) \( \implies \) (d). For (d) \( \implies \) (e) note that \( A^*(I) \subseteq \bigcup_{n \geq 1} \text{Ass}(A/I^n) = \bigcup_{n \geq 1} \text{Ass}(A/T^n) = \overline{A^*(I)} \subseteq A^*(I) \), hence \( A^*(I) = \overline{A^*(I)} \). For the implication (e) \( \implies \) (a) we refer to the original paper of Ratliff [17, Theorem 4.3] or [15, Theorem 8.12].

3. Symbolic Powers of Ideals

In this section we obtain several consequences regarding the behavior of the symbolic powers of a prime ideal. We begin with a well-known observation that will be used extensively.

**Remark 3.1.** Let \( A \) be a locally formally equidimensional ring and \( I \) an ideal in \( A \). The following are equivalent:

(a) \( \ell(I_A q) < \dim A_q \) for every prime \( q \in V(I) \setminus \text{Min}(A/I); \)
(b) $\mathcal{A}^\ast(I) = \text{Min}(A/I)$.

The equivalence follows from a well-known result of McAdam that states that $q \in \mathcal{A}^\ast(I)$ if and only if $\ell(IA_q) = \dim A_q$ ([15, Proposition 4.1]). The importance of the condition (a) in the context of symbolic powers of ideals has been documented in many instances (e.g. [7,8,10]). If $I = p$ is a prime in a locally formally equidimensional ring $A$, the condition also implies that the $p$-adic topology and the $p$-symbolic topology on $A$ are linearly equivalent, i.e. there exists $k$ such that $p^{(n+k)} \subseteq p^n$ for all $n \geq 1$ ([18, Corollary 1]). We also note that if $I$ is an equimultiple ideal in a local ring, then $I$ satisfies the condition (a).

Recall that for an ideal $I$ in a noetherian ring $A$, the unmixed part $I^{\text{unm}}$ is the intersection of the primary components of $I$ that correspond to the minimal prime ideals over $I$. For $n \geq 1$ and a prime ideal $p$, the ideal $(p^n)^{\text{unm}}$ is referred to as the $n$-th symbolic power of $p$ and is typically denoted by $p^{(n)}$.

We now state a non-local version of a result we proved in [5].

**Proposition 3.2.** Let $A$ be a locally formally equidimensional domain and $I$ an ideal in $A$ such that $\ell(IA_q) < \dim A_q$ for every prime $q \in V(I) \setminus \text{Min}(A/I)$. Assume that the extended Rees algebra $R = A[It, t^{-1}]$ has an $S_2$ification $S_2(R) = \bigoplus_{n \in \mathbb{Z}} I_nt^n$. Then, for $n \geq 1$,

$$(I^n)^{\text{unm}} \subseteq S_2(I^n).$$

**Proof.** Since $\ell(IA_q) = \ell(I^nA_q)$ and the $n$-th Veronese subring of $S_2(R)$ is an $S_2$-ification of the $n$-th Veronese subring of $R$, it is enough to prove the conclusion for $n = 1$. Furthermore, it is enough to prove that for every prime $p$ that contains $I$ we have $(I_p)^{\text{unm}} \subseteq (I_1)_p$. We may also assume that $p \notin \text{Min}(A/I)$, for otherwise $(I_p)^{\text{unm}} = I_p$ and the inclusion is clear. We now note that if $p \in \text{Spec}(R)$, then $\bigoplus_{n \in \mathbb{Z}} (I_n)_pt^n$ is an $S_2$-ification of $A_p[I_pt, t^{-1}]$. Hence, after localizing at $p$, we may assume that $(A, m)$ is a formally equidimensional ring and $I$ is an ideal in $A$ such that $\ell(IA_q) < \dim A_q$ for every prime $q \in V(I) \setminus \text{Min}(A/I)$; we need to show that $I^{\text{unm}} \subseteq I_1 \cap A$. This was proved in [5, Lemma 3.10]. Even though the statement of [5, Lemma 3.10] assumes that $I$ is equimultiple, the proof given there works for any ideal $I$ such that $\ell(IA_q) < \dim A_q$ for every prime $q \in V(I) \setminus \text{Min}(A/I)$. Also, in [5], the additional
conditions on the ring were imposed just to guarantee the existence of the $S_2$-ification of the extended Rees algebra. □

Remark 3.3. In the previous proposition, the requirement that $A$ be a domain was imposed just to ensure that we are in the setup used in the description of the $S_2$-ification process in (2.12) where we construct the $S_2$-ification $S_2(R)$ inside the quotient field $Q(R)$. If the extended Rees algebra $R$ already satisfies the $(S_2)$ property, a particular case in which Proposition 3.2 will be applied subsequently, there is no need to assume that $A$ is a domain.

We now consider a prime ideal $p$ that satisfies the equivalent conditions in Remark 3.1. Under mild conditions on the ring $A$, the next result shows that if the associated graded ring $G_{p,A_p}(A_p)$ satisfies $(S_1)$, then the filtration of symbolic powers coincides with the $S_2$-filtration $S_2(p^n)$. As discussed in more detail at the end of the proof of this result, previous results in the literature established that $p^{(n)} = p^n$ for all $n$ if $G_p(A)$ satisfies $(S_1)$, and $p^{(n)} = \overline{p^n}$ for all $n$ if $G_{p,A_p}(A_p)$ is reduced.

Theorem 3.4. Let $A$ be a locally formally equidimensional domain and $p$ a prime ideal in $A$ such that the extended Rees algebra $A[pt, t^{-1}]$ has an $S_2$-ification. Assume that $G_{p,A_p}(A_p)$ satisfies $(S_1)$. The following are equivalent:

(a) $A^*(p) = \{p\}$;
(b) $p^{(n)} = S_2(p^n)$ for all $n \geq 1$;
(c) $p^{(n)} = S_2(p^n)$ for infinitely many $n$.

Proof. We first prove $(a) \Rightarrow (b)$. By Remark 3.1 we can apply Proposition 3.2 and obtain that $p^{(n)} \subseteq S_2(p^n)$. To show $S_2(p^n) \subseteq p^{(n)}$, it is enough to prove that $S_2(p^n)A_p \subseteq p^nA_p$. To see this, note that since $G_{p,A_p}(A_p)$ satisfies $(S_1)$, the extended Rees algebra $A_p[pA_p t, t^{-1}]$ satisfies $(S_2)$ (Remark 2.16), so $S_2(p^nA_p) = p^nA_p$. As $S_2(p^nA_p) = S_2(p^n)A_p$, the inclusion follows.

We now prove $(c) \Rightarrow (a)$. For $n$ large enough, by Corollary 2.15 we have

$$A^*(p) = \text{Ass}(A/S_2(p^n)) = \text{Ass}(A/p^{(n)}) = \{p\}.$$
The implication \((a) \Rightarrow (c)\) of the next Corollary recovers a result of Huckaba [8, Theorem 2.1]. His result, even though it was stated under the assumption \(G_p(A)\) Cohen-Macaulay, has a proof that only requires that \(G_p(A)\) satisfy \((S_1)\).

**Corollary 3.5.** Let \(A\) be a locally formally equidimensional ring and \(p\) a prime ideal in \(A\). Assume that \(G_p(A)\) satisfies the \((S_1)\) property. The following are equivalent:

\begin{enumerate}[(a)]
  \item \(\overline{A}^*(p) = \{p\}\);
  \item \(A^*(p) = \{p\}\);
  \item \(p^{(n)} = p^n\) for all \(n \geq 1\);
  \item \(p^{(n)} = p^n\) for infinitely many \(n\).
\end{enumerate}

**Proof.** Since \(G_p(A)\) satisfies \((S_1)\), the extended Rees algebra \(A[pt, t^{-1}]\) satisfies \((S_2)\). For \((a) \Rightarrow (c)\), note that, by Proposition 3.2 and Remark 3.3, we have \(p^{(n)} \subseteq p^n\) for every \(n \geq 1\), and hence equality holds. To see \((d) \Rightarrow (b)\), note that for \(n\) large enough we have \(A^*(p) = \text{Ass}(A/p^n) = \text{Ass}(A/p^{(n)}) = \{p\}\). Finally, as \(\overline{A}^*(p) \subseteq A^*(p)\), the implication \((b) \Rightarrow (a)\) also follows. \(\square\)

**Remark 3.6.** Let \(A\) be a Cohen-Macaulay ring and \(p\) a prime ideal of height \(h > 0\) such that \(A_p\) is regular and \(p\) is generated by \(h + 1\) elements. By [3, Proposition 2.6], the Rees algebra \(A[pt]\) satisfies \((S_2)\), hence \(G_p(A)\) satisfies \((S_1)\) (Remark 2.16), so the conclusions of Corollary 3.5 follow. This recovers results from [2, 11]. For the local case, Huneke and Huckaba [9, Theorem 2.5] obtained the same conclusions of Corollary 3.5 under the weaker assumption that the analytic spread \(\ell(p) = h + 1\).

**Remark 3.7.** Let \(A\) be a locally formally equidimensional domain and \(p\) a prime ideal in \(A\). Assume that \(G_{p,A_p}(A_p)\) is reduced. By a result of Huckaba [7, Theorem 1.4] (which extends work of Huneke [10, Theorem 2.1]), the following are equivalent:

\begin{enumerate}[(a)]
  \item \(\overline{A}^*(p) = \{p\}\);
  \item \(p^{(n)} = \overline{p^n}\) for all \(n \geq 1\).
\end{enumerate}

Since \(G_{p,A_p}(A_p)\) is reduced, it satisfies the \((S_1)\) property. Therefore, by Theorem 3.4 the above equivalent conditions imply that \(S_2(p^n) = \overline{p^n}\) for all \(n \geq 1\). If \(A\) is a normal domain, this shows that the extended Rees algebra \(A[pt, t^{-1}]\) is regular in codimension one.
Indeed, if \( Q \) is a height one prime ideal in \( A[pt, t^{-1}] \), then \( A[pt, t^{-1}]_Q \cong S_2(A[pt, t^{-1}]_Q) \cong S_2(A[pt, t^{-1}])_Q \) which is a local integrally closed ring because \( S_2(A[pt, t^{-1}]) \) coincides with the integral closure of \( A[pt, t^{-1}] \).

The following example is a modification of [7, Example 1.6]. It describes a situation where all the hypotheses of Theorem 3.4 are satisfied for an equimultiple prime ideal \( p \), and hence \( p^{(n)} = S_2(p^n) \) for all \( n \geq 1 \). On the other hand, the associated graded ring \( G_p(A) \) does not satisfy the \((S_1)\) property.

**Example 3.8.** Let \( A = k[u, v, w, z, r]_{(u,v,w,z,r)}/(w^7 - u^{35}z^2 + u^{30}v, v^3 - wz) \) and the prime ideal \( p = (u, v, w, r)A \). Using Macaulay2 [13] one can check that \( A \) is a three dimensional Cohen-Macaulay domain and \( \text{ht} \ p = \ell(p) = 2 \), so \( p \) is an equimultiple ideal. This also implies that \( \mathcal{T}(p) = \{p\} \). The associated graded ring \( G_p(A) \) has a unique minimal prime ideal \((v^*, w^*)G_p(A)\), where \( v^*, w^* \in p/p^2 \subseteq G_p(A) \) are the images of \( v \) and \( w \), respectively. Moreover, the associated prime ideals of \( G_p(A) \) are \((v^*, w^*)G_p(A)\) and \((v^*, w^*, z^*)G_p(A)\), and hence \( G_p(A) \) does not satisfy Serre’s \((S_1)\) property.

In the local ring \( A_p \) we have \( w = (1/z)v^3 \), so \( w^7 - u^{35}z^2 + u^{30}v = (1/z^7)(v^{21} - u^{35}z^9 + u^{30}vz^7) \) and therefore

\[
A_p \cong k[u, v, z, r]_{(u,v,r)}/(v^{21} - u^{35}z^9 + u^{30}vz^7).
\]

Since \( A_p \) is a hypersurface \( R/(f) \), where \( R = k[u, v, z, r]_{(u,v,r)} \) and \( f = v^{21} - u^{35}z^9 + u^{30}vz^7 \) with initial term \( f^* = v^{21} \), it follows that \( G_{pA_p}(A_p) \cong k[u, v, z, r]_{(u,v,r)}/(v^{21}) \). Note that \( G_{pA_p}(A_p) \) is not reduced, but it has a unique minimal prime ideal and no embedded associated prime ideals, so it satisfies the \((S_1)\) property. This implies that the extended Rees algebra \( A_p[pA_p, t, t^{-1}] \) satisfies the \((S_2)\) property, and therefore \( S_2(p^nA_p) = p^nA_p \) for all \( n \geq 1 \).

On the other hand, let us note that the ideal \( pA_p \) is not normal, i.e. not all of its powers are integrally closed. This follows from the criterion for normality of the maximal ideal in a hypersurface that is proved in [3, 2.4]. In fact, in the ring \( A \) one can check with Macaulay2 that \( w^7 \in p^{28} \) and \( w \notin p^2 \). Then \( w \in p^2 \subseteq p^3 \), hence \( p^2 \neq p^3 \). Using the procedure outlined in [4, Proposition 3.2] (which is valid in any affine domain), by identifying the \( S_2 \)-ification of the Rees algebra \( A[pt] \) with the ring of endomorphisms of the canonical ideal of \( A[pt] \), after
lengthy computations with Macaulay2 we were able to obtain

\[ S_2(p^2) = p^2 + (w), \]
\[ S_2(p^3) = p^3 + (w) \supseteq pS_2(p^2), \]
\[ S_2(p^4) = p^4 + (wr, w^2, vw, uw) = pS_2(p^3) + (S_2(p^2))^2, \text{ and} \]
\[ S_2(p^5) = p^5 + (w^2, vr, v^2w, uvw, u^2w, uvr) = pS_2(p^4) + S_2(p^2)S_2(p^3). \]

One can also check that \( w / \not\in S_2(p^4). \) As noticed before, \( w \in p^4 \setminus p^2, \) so we have the strict inclusions

\[ p^4 \subsetneq S_2(p^4) \subsetneq p^4. \]

On the other hand, since \( G_{pA_p}(A_p) \) satisfies \((S_1),\) by Theorem 3.4 we know that \( p^{(n)} = S_2(p^n) \) for all \( n \geq 1. \) In fact, in this particular example, for \( i \in \{2, 3, 4, 5\} \) we have \( \text{Ass}(A/p^i) = \{p, m\}, \) where \( m = (u, v, w, z, r)A, \) so \( p^{(i)} = (p^i : m^\infty). \) Compared to \( S_2(p^i), \) the saturations \( (p^i : m^\infty) \) are much easier to compute in Macaulay2 and we double checked that the ideals \( (p^i : m^\infty) \) coincide with the ideals \( S_2(p^i) \) \((2 \leq i \leq 5) \) computed above.

4. Other Applications

**Proposition 4.1.** Let \((A, m)\) be a formally equidimensional local ring and \( I \) an ideal of positive height and analytic spread \( \ell(I) < \dim A. \) Assume that \( A[It, t^{-1}] \) satisfies the \((S_2)\) property. Then

\[ (I^n : m^\infty) = I^n \text{ for all } n \geq 1, \]

i.e., \( m \not\in \bigcup_{n \geq 1} \text{Ass}(A/I^n). \)

**Proof.** By Corollary 2.10 we have \( \overline{A^*}(I) = \bigcup_{n \geq 1} \text{Ass}(A/I^n) \) and \( m \in \overline{A^*}(I) \) if and only if \( \ell(I) = \dim A \) ([15 Proposition 4.1]). \qed

**Remark 4.2.** If the ring \( A \) is also a domain, by using very different methods we already obtained the above result in [5, Corollary 3.15].

We are also able to recover the following result of Brodmann regarding the asymptotic primes of an almost complete intersection ideal.
Proposition 4.3. [2, Proposition 3.9] Let $A$ be a Cohen-Macaulay ring and $I$ an ideal of height $h$ that can be generated by $h + 1$ elements. Moreover, assume that $I$ is generically a complete intersection. Then

(a) $\text{Ass}(A/I^n) \subseteq \text{Ass}(A/I^{n+1})$ for all $n \geq 1$;
(b) $\text{ht } p \leq h + 1$ for all $p \in A^*(I)$.

Proof. Note that we may assume $h \geq 1$, for otherwise $I = (0)$ and the conclusions clearly follow. Under the given assumptions on the ideal $I$, by [3, Proposition 2.6], the Rees algebra $A[It]$ satisfies $(S_2)$. Then $A[It, t^{-1}]$ satisfies $(S_2)$ as well (Remark [2,16]), and (a) follows from the first part of Corollary [2,10]. By the same Corollary [2,10], we note that $A^*(I) = \bigcup_{n \geq 1} \text{Ass}(A/I^n) = \bigcup_{n \geq 1} \text{Ass}(A/I^n) = \overline{A}^*(I)$. Then, for $p \in A^*(I) = \overline{A}^*(I)$, we have $\text{ht } p = \ell(I_p)$ ([15, Proposition 4.1]). Since $I$ is generated by $h + 1$ elements, we also have $\ell(I_p) \leq \mu(I_p) \leq h + 1$, and part (c) follows.

Using a result of Zarzuela [20], we obtain a similar conclusion for certain ideals of analytic deviation one.

Proposition 4.4. Let $(A, m)$ be a local Cohen-Macaulay ring with infinite residue field and $I$ an ideal of height $h \geq 1$ and analytic spread $\ell(I) = h + 1$. Assume that $I$ is generically a complete intersection, the reduction number $r(I)$ is at most one, and $A/I$ satisfies $(S_1)$. Then

(a) $\text{Ass}(A/I^n) \subseteq \text{Ass}(A/I^{n+1})$ for all $n \geq 1$;
(b) $\text{ht } p \leq h + 1$ for all $p \in A^*(I)$.

Proof. By [20, Theorem 4.4], the associated graded ring $G_I(A)$ satisfies $(S_1)$, or equivalently, $A[It, t^{-1}]$ satisfies $(S_2)$. Then part (a) follows from the first part of Corollary [2,10]. For part (b), as in the proof of Proposition 4.3 we note that $A^*(I) = \overline{A}^*(I)$ and for $p \in \overline{A}^*(I)$ we have $\text{ht } p = \ell(I_p) \leq \ell(I) = h + 1$.

In the case of a prime ideal in a regular local ring, we record the following Corollary.

Corollary 4.5. Let $(A, m)$ be a local Cohen-Macaulay ring and $p$ a prime ideal of height $h$ such that $A_p$ is a regular local ring. Assume that either
(i) \( \mu(p) = h + 1 \) or
(ii) \( A/\mathfrak{m} \) is infinite, \( \ell(p) = h + 1 \) and \( r(p) \leq 1 \).

Then:

(a) \( \text{Ass}(A/p^n) \subseteq \text{Ass}(A/p^{n+1}) \) for all \( n \geq 1 \);
(b) \( \text{ht} q \leq h + 1 \) for all \( q \in A^*(p) \).

References


Department of Mathematics 2750, North Dakota State University, PO BOX 6050, Fargo, ND 58108-6050, USA

*E-mail address: catalin.ciuperca@ndsu.edu*