REDUCTION NUMBERS OF EQUIMULTIPLE IDEALS

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Abstract. Let $(A, \mathfrak{m})$ be an unmixed local ring containing a field. If J is an $\mathfrak{m}$-primary ideal with Hilbert-Samuel multiplicity $e_A(J)$, a recent result of Hickel shows that every element in the integral closure $\overline{J}$ satisfies an equation of integral dependence over J of degree at most $\mathrm{e}_A(J)$. We extend this result to equimultiple ideals J by showing that the degree of such an equation of integral dependence is at most $c_q(J)$, where $c_q(J)$ is one of the elements of the so-called multiplicity sequence introduced by Achilles and Manaresi. As a consequence, if the characteristic of the field contained in A is zero, it follows that the reduction number of an equimultiple ideal J with respect to any minimal reduction is at most $c_q(J) - 1$.

1. Introduction

Let $(A, \mathfrak{m})$ be local noetherian ring and J a proper ideal of A. An element $x \in A$ is said to be integral over J if it satisfies an equation of integral dependence

$$x^n + a_1 x^{n-1} + \cdots + a_n = 0$$

with coefficients $a_i \in J^i$. Equivalently, $(J + xA)^n = J(J + xA)^{n-1}$ for some positive integer n. The elements that are integral over J form an ideal which we denote $\overline{J}$.

If J is an $\mathfrak{m}$-primary ideal in an unmixed local ring A that contains a field, a recent result of Hickel [12, Theorem 1.1] shows that every element of $\overline{J}$ satisfies an equation of integral dependence over J of degree at most $\mathrm{e}_A(J)$, the Hilbert-Samuel multiplicity of J. The technique used by Hickel was introduced by Scheja in [20] and reduces the problem to the hypersurface case by using the fact that a complete local ring containing a field is a finite extension of a formal power series over a coefficient field. The technique was previously used to provide much simplified proofs for the theorem of Rees that states that two $\mathfrak{m}$-primary ideals $J \subseteq I$ form a reduction (equivalently, $I \subseteq \overline{J}$) if and only if I and J have the same

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Hilbert-Samuel multiplicity (see [20] and [13, Discussion 11.3.3]). It was also used in [16] to give a simplified proof in the case when the ring contains a field for the fact that an unmixed local ring of multiplicity one must be regular. It should be noted that the last two results that we just mentioned were already known to be true without assuming that the ring contains a field, but the proofs were much more involved. In contrast to this, as far as this author knows, Hickel's result is new and there are no known proofs for the case when the ring does not contain a field.

In this paper we consider the more general case when the ideal J is equimultiple, i.e. J satisfies the condition $\ell(J) = \operatorname{ht} J$, where $\ell(J)$ is the analytic spread of J and $\operatorname{ht} J$ is the height of J. Because of their geometric interpretations, these ideals have been extensively studied. We refer the reader to the book [11] by Herrmann, Ikeda, and Orbanz for a detailed account of equimultiple ideals from an algebraic perspective.

If A is a local unmixed ring that contains a field, we show in Theorem 3.2 that an element in the integral closure of an equimultiple ideal J satisfies an equation of integral dependence of degree at most $c_q(J)$, which is one of the numerical invariants in the multiplicity sequence introduced by Achilles and Manaresi (see 2.5). We note here that $c_q(J)$ agrees with the Hilbert-Samuel multiplicity $\mathrm{e}_A(J)$ when J is $\mathfrak{m}$-primary. The proof relies again on Scheja's technique and, up to a point, follows Hickel's argument in the $\mathfrak{m}$-primary case. The core and technical part of the proof, which is an argument needed only in the case when the ideal J is not $\mathfrak{m}$-primary, is extracted conveniently in Lemma 3.1. As an immediate consequence, if the characteristic of the field contained in A is zero, Corollary 3.4 shows that $r_K(J) \leq c_q(J) - 1$, where $r_K(J)$ is the reduction number of J with respect to any minimal reduction K.

Bounds on the reduction number of an ideal J play an important role in the study of the Cohen-Macaulay property of the Rees algebra and the associated graded ring of J. Results that give an upper bound on the reduction number of an ideal were previously known in the case of $\mathfrak{m}$-primary ideals in Cohen-Macaulay rings. If $d = \dim A$, Sally [19, 2.2.] first proved that $r(\mathfrak{m}) \leq d!\,\mathrm{e}_A(\mathfrak{m}) - 1$. This was later improved by Vasconcelos [21, 6.12, 6.16] who showed that $r(J) \leq d\,\mathrm{e}_A(J) - 2d + 1$ for every $\mathfrak{m}$-primary ideal J. Sharper bounds for the reduction number were also known under stronger assumptions that involved various graded algebras associated with the ideal. Among those, we note the bound $r(J) \leq d - 1$ obtained by Goto and Shimoda ([9]) in the case when the Rees algebra of J is Cohen-Macaulay. We refer the reader to [8, 18, 23] for other bounds of this type. For equimultiple ideals J, Grothe, Herrmann and Orbanz [10] generalized the Goto-Shimoda bound and showed that if the Rees algebra of J is Cohen-Macaulay, then $r(J) \leq \operatorname{ht} J - 1$. However, without assuming the Cohen-Macaulayness of the Rees algebra, it should be noted that for $d > 1$ there are examples that show that one cannot obtain bounds for $r(J)$ that only depend on the ring A even when A is a regular local ring.

2. Background

Throughout this paper all rings are commutative, noetherian, and have an identity. If $(A, \mathfrak{m})$ is a local noetherian ring with maximal ideal $\mathfrak{m}$, we say that A is formally equidimensional if all the minimal prime ideals of the completion $\widehat{A}$ have the same dimension. We also say that A is unmixed if all the associated prime ideals of the completion $\widehat{A}$ have the same dimension. Hence A is unmixed if and only if A is formally equidimensional and $\widehat{A}$ has no embedded prime ideals.

In this section we present several definitions and results that will be needed in the paper. For terminology not otherwise explained, we refer the reader to [13].

2.1 (Reductions, analytic spread, and equimultiple ideals)**.** If I is a proper ideal in the local ring A and $\mathcal{F} = \bigoplus_{n \geq 0} I^n/\mathfrak{m}I^n$ is the fiber cone of I, the analytic spread of I is defined by $\ell(I) = \dim \mathcal{F}$. One has the inequalities

$$\operatorname{ht} I \leq \operatorname{alt} I \leq \ell(I) \leq \dim A, \tag{2.1.1}$$

where $\operatorname{ht} I$ is the height of the ideal I and $\operatorname{alt} I = \max\{\operatorname{ht} \mathfrak{p} \mid \mathfrak{p} \in \operatorname{Min}(A/I)\}$. An ideal with $\ell(I) = \operatorname{ht} I$ is called an equimultiple ideal.

If $J \subseteq I$, we say that J is a reduction of I if $JI^n = I^{n+1}$ for some non-negative integer n. The smallest n with this property, denoted $r_J(I)$, is called the reduction number of I with respect to J. The reductions of I that are minimal with respect to inclusion are called minimal reductions and, if the residue field $A/\mathfrak{m}$ is infinite, it is known that all the minimal reductions of I are minimally generated by $\ell(I)$ elements. The reduction number

$r(I)$ is defined to be the smallest reduction number of I with respect to a minimal reduction. Similarly, the big reduction number $br(I)$ is the largest reduction number of I with respect to a minimal reduction. We note that $br(I)$ is always finite; see for example [22, Section 2].

2.2 (The Hilbert-Samuel multiplicity)**.** If $(A, \mathfrak{m})$ is a local noetherian ring, I is an $\mathfrak{m}$-primary ideal, and M is a finitely generated A-module of dimension $d = \dim M$, the multiplicity $\mathrm{e}_A(I, M)$ is defined to be the normalized leading coefficient of the Hilbert function

$$\lambda_A(M/I^nM) = \frac{\mathrm{e}_A(I, M)}{d!} n^d + \text{lower degree terms} \quad (n \gg 0).$$

We note that some texts (e.g. [13]) define $\mathrm{e}_A(I, M)$ by using $d = \dim A$, in which case the multiplicity of M is non-zero if and only if M and A have the same dimension. When $M = A$ we simply write $\mathrm{e}_A(I)$ instead of $\mathrm{e}_A(I, A)$. If $I = \mathfrak{m}$, we also write $\mathrm{e}_A(M)$ when we mean $\mathrm{e}_A(\mathfrak{m}, M)$.

2.3 (The associativity formula; (24.7) in [15])**.** Let $(A, \mathfrak{m})$ be a local noetherian ring and $x_1, \ldots, x_d$ a system of parameters ($d = \dim A$). Let t be a positive integer with $1 \leq t \leq d$. If we set

$$\Lambda = \{\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Spec} A, \mathfrak{p} \supseteq (x_1, \ldots, x_t), \operatorname{ht} \mathfrak{p} = t, \dim A/\mathfrak{p} = d - t\},$$

then

$$\mathrm{e}_A(x_1, \ldots, x_d) = \sum_{\mathfrak{p} \in \Lambda} \mathrm{e}_A((x_{t+1}, \ldots, x_d), A/\mathfrak{p})\, \mathrm{e}_{A_\mathfrak{p}}((x_1, \ldots, x_t)A_\mathfrak{p}).$$

2.4 (Multiplicity and rank)**.** If $(A, \mathfrak{m}) \hookrightarrow (B, \mathfrak{n})$ is a finite local extension of integral domains and I is an $\mathfrak{m}$-primary ideal, then

$$\operatorname{rank}_A B \cdot \mathrm{e}_A(I, A) = \mathrm{e}_A(I, B) = \mathrm{e}_B(IB, B) \cdot [B/\mathfrak{n} : A/\mathfrak{m}].$$

This is a particular case of [13, 11.2.6 and 11.2.7].

2.5 (The multiplicity sequence)**.** Let $(A, \mathfrak{m})$ be a local noetherian ring of dimension d and I an ideal in A. If I is not necessarily $\mathfrak{m}$-primary, many papers in the literature deal with the problem of generalizing the classical multiplicity $\mathrm{e}_A(I)$. Most relevant to our discussion is the introduction of the so-called generalized multiplicity sequence $c_0(I), \ldots, c_d(I)$ by Achilles and Manaresi [2]. In brief, if $k = A/\mathfrak{m}$ consider the k-algebra $G_\mathfrak{m}(G_I(A)) = \bigoplus_{i,j \geq 0} (\mathfrak{m}^i I^j +$

$I^{j+1})/(\mathfrak{m}^{i+1}I^j+I^{j+1})$, let $h(i,j)=\dim_k(\mathfrak{m}^iI^j+I^{j+1})/(\mathfrak{m}^{i+1}I^j+I^{j+1})$ be its bigraded Hilbert function, and let $H(i,j):=\sum_{u=0}^{i}\sum_{v=0}^{j}h(u,v)$ be the double sum transform of $h(i,j)$. For $i,j\gg 0$ the function $H(i,j)$ becomes a polynomial function of degree d

$$p(i,j)=\sum_{\substack{k,l\geq 0\\ k+l\leq d}} a_{k,l}\binom{i}{k}\binom{j}{l}.$$

The multiplicity sequence of Achilles and Manaresi is defined by $c_k(I):=a_{k,d-k}$ for $k=0,\ldots,d$. It is known that the first element $c_0(I)$ coincides with the j-multiplicity $j(I)$ introduced earlier by the same authors in [1]. Moreover, if I is $\mathfrak{m}$-primary, then $c_0(I)=\mathrm{e}_A(I)$ and $c_k(I)=0$ for all $k>0$. We refer the reader to [2] and the expository paper [3] for a detailed account of these coefficients and their properties. We also note that many results in the literature deal with the problem of using these invariants to describe numerically an arbitrary ideal I the same way the Hilbert-Samuel multiplicity characterizes or gives information about an $\mathfrak{m}$-primary ideal.

If the ideal I is equimultiple, which is the case of interest in our paper, and $q:=\dim A/I$, then $c_i(I)=0$ for $i\neq q$ and

$$c_q(I)=\sum_{\mathfrak{p}\in\Lambda}\mathrm{e}_A(A/\mathfrak{p})\cdot\mathrm{e}_{A_\mathfrak{p}}(IA_\mathfrak{p}), \tag{2.5.1}$$

where $\Lambda=\{\mathfrak{p}\in\operatorname{Spec}A\mid\mathfrak{p}\supseteq I,\dim A/\mathfrak{p}=\dim A/I,\dim A/\mathfrak{p}+\operatorname{ht}\mathfrak{p}=\dim A\}$ (see [2, Proposition 2.3]). Note that Λ is always non-void (and hence $c_q(I)>0$) if A is formally equidimensional.

The coefficient $c_q(I)$ also satisfies the so-called linearity formula:

$$c_q(I)=\sum_{\mathfrak{p}}c_q(I(A/\mathfrak{p}))\lambda(A_\mathfrak{p}), \tag{2.5.2}$$

where the sum runs over all the prime ideals $\mathfrak{p}$ with $\dim A/\mathfrak{p}=\dim A$. If I is $\mathfrak{m}$-primary, this is the linearity formula for the Hilbert-Samuel multiplicity. The general case can be obtained by applying (2.5.1) in conjunction with the linearity formula for $\mathfrak{m}$-primary ideals.

2.6 (Multiplicities and reductions)**.** Let $(A,\mathfrak{m})$ be a formally equidimensional local ring and $I\subseteq J$ proper ideals. If I is $\mathfrak{m}$-primary, a well known result of Rees shows that $I\subseteq J$ is a reduction (equivalently, $J\subseteq\overline{I}$) if and only if $\mathrm{e}_A(I)=\mathrm{e}_A(J)$. If I and J are not necessarily

$\mathfrak{m}$-primary, but I is equimultiple and $\sqrt{I} = \sqrt{J}$, a criterion proved by Böger [4] shows that $I \subseteq J$ is a reduction if and only if $\mathrm{e}_{A_\mathfrak{p}}(IA_\mathfrak{p}) = \mathrm{e}_{A_\mathfrak{p}}(JA_\mathfrak{p})$ for all the prime ideals $\mathfrak{p}$ minimal over I. In view of (2.5.1), this criterion can be restated as follows: if I equimultiple and $I \subseteq J \subseteq \sqrt{I}$, then $I \subseteq J$ is a reduction if and only if $c_q(I) = c_q(J)$. In many ways, the coefficient $c_q(I)$ of an equimultiple ideal I plays the same role as the classical Hilbert-Samuel multiplicity of an $\mathfrak{m}$-primary ideal. See also Proposition 2.7 below.

For arbitrary ideals $I \subseteq J \subseteq \sqrt{I}$, it is also known that if $I \subseteq J$ is a reduction, then $c_i(I) = c_i(J)$ for all $i = 0, \ldots, d$ [6, 2.7]. As far as this author knows, there is no proof in the literature for the converse of this result in an arbitrary formally equidimensional local ring.

If I is an $\mathfrak{m}$-primary ideal in an unmixed local ring $(A, \mathfrak{m})$, a classical result of Nagata ([15, 40.6]) shows that $\mathrm{e}_A(I) = 1$ if and only if $I = \mathfrak{m}$ and A is a regular local ring. More generally, if I is an equimultiple ideal, the following result shows that a similar characterization of regular local rings can be given by using the invariant $c_q(I)$.

Proposition 2.7. *Let $(A, \mathfrak{m})$ be an unmixed local ring and I an equimultiple ideal of A of dimension $q = \dim A/I$. Then $c_q(I) = 1$ if and only if A is a regular local ring and I is (a prime ideal) generated by a part of a regular system of parameters.*

Proof. If A is a regular local ring, $x_1, \ldots, x_d$ is a regular system of parameters, and $\mathfrak{p} = (x_1, \ldots, x_l)$, then $A/\mathfrak{p}$ and $A_\mathfrak{p}$ are regular local rings and from (2.5.1) we obtain

$$c_q(\mathfrak{p}) = \mathrm{e}_A(A/\mathfrak{p})\, \mathrm{e}_{A_\mathfrak{p}}(\mathfrak{p}A_\mathfrak{p}) = 1.$$

We now assume that $c_q(I) = 1$. By passing to the completion $\widehat{A}$, we may assume that A is a complete equidimensional local ring with $\mathrm{Ass}(A) = \mathrm{Min}(A)$. Since I is equimultiple, all the minimal prime ideals over I have the same height (see 2.1.1). Then we have $1 = c_q(I) = \sum_\mathfrak{p} \mathrm{e}(A/\mathfrak{p})\, \mathrm{e}_{A_\mathfrak{p}}(I_\mathfrak{p})$, where the sum runs over all the minimal primes $\mathfrak{p}$ over I such that $\dim A/\mathfrak{p} = \dim A/I$, so there is a unique minimal prime ideal $\mathfrak{p}$ over I and $\mathrm{e}_A(A/\mathfrak{p}) = \mathrm{e}_{A_\mathfrak{p}}(I_\mathfrak{p}) = 1$. This implies that $A/\mathfrak{p}$ is a regular ring, $IA_\mathfrak{p} = \mathfrak{p}A_\mathfrak{p}$, and $A_\mathfrak{p}$ is a regular local ring. On the other hand, since I is equimultiple, $\overline{I}$ has no embedded prime ideals [17, 2.12], so $\mathrm{Ass}(A/\overline{I}) = \{\mathfrak{p}\}$. But $\overline{I}A_\mathfrak{p} = \mathfrak{p}A_\mathfrak{p}$, so we must have $\overline{I} = \mathfrak{p}$. From this we obtain $\ell(\mathfrak{p}) = \ell(I) = \mathrm{ht}\, I = \mathrm{ht}\, \mathfrak{p}$, so $\mathfrak{p}$ is also an equimultiple ideal. Now let $x_1, \ldots, x_l$ be a minimal reduction of I (hence a minimal reduction of $\mathfrak{p}$, too) and let $x_{l+1}, \ldots, x_d$ be a regular system of parameters of $A/\mathfrak{p}$. Then

$$\begin{aligned}
\mathrm{e}_A(\mathfrak{m}, A) &= \mathrm{e}_A(\mathfrak{p} + (x_{l+1}, \ldots, x_d), A) = \mathrm{e}_A((x_1, \ldots, x_l, x_{l+1}, \ldots, x_d), A) \\
&= \mathrm{e}_A((x_{l+1}, \ldots, x_d), A/\mathfrak{p})\, \mathrm{e}_{A_\mathfrak{p}}((x_1, \ldots, x_l)A_\mathfrak{p}) \qquad \text{(cf. 2.3)} \\
&= 1 \cdot \mathrm{e}_{A_\mathfrak{p}}(IA_\mathfrak{p}) = 1,
\end{aligned}$$

so A is a regular local ring. As $A/\mathfrak{p}$ is also a regular local ring, it follows that $\mathfrak{p}$ is generated by elements that are part of a regular system of parameters of A. Finally, since I is a reduction of $\mathfrak{p}$, we must have $I = \mathfrak{p}$, finishing the proof. □

3. The degree of an equation of integral dependence

Some technical aspects in the proof of the main result (Theorem 3.2), which only occur for ideals that are not $\mathfrak{m}$-primary, are collected in the following Lemma.

Lemma 3.1. *Let k be a field, $T = k[[X_1, \ldots, X_l, X_{l+1}, \ldots, X_d, Y]]$ and let*

$$B = T/(f) = k[[x_1, \ldots, x_l, x_{l+1}, \ldots, x_d, y]],$$

where

$$f = Y^r + a_1(X_1, \ldots, X_d)Y^{r-1} + \cdots + a_r(X_1, \ldots, X_d)$$

with $r \geq 1$ and $a_i(X_1, \ldots, X_d) \in k[[X_1, \ldots, X_d]]$ for all $i \in \{1, \ldots, r\}$. Let $P = (X_1, \ldots, X_l, Y)T$ and $\mathfrak{p} = PB = (x_1, \ldots, x_l, y)B$. Assume that $d \geq l + 1$ and $\operatorname{ht} \mathfrak{p} = \ell(\mathfrak{p}) = l \geq 1$. Then the following hold:

(a) $\mathfrak{p}$ *is a prime ideal;*
(b) $a_r(0, \ldots, 0, X_{l+1}, \ldots, X_d) = 0$ *and* $f \in (X_1, \ldots, X_l, Y)T$*;*
(c) $B/\mathfrak{p} \cong k[[X_{l+1}, \ldots, X_d]]$*;*
(d) $\operatorname{Ass}(B/\mathfrak{p}^n) = \{\mathfrak{p}\}$ *for all* $n \geq 1$*;*
(e) $x_{l+1}, \ldots, x_d$ *is a regular sequence on* $B/\mathfrak{p}^n$ *for all* $n \geq 1$*;*
(f) $(x_{l+1}, \ldots, x_d) \cap \mathfrak{p}^n = (x_{l+1}, \ldots, x_d)\mathfrak{p}^n$ *for all* $n \geq 1$.

Proof. We have

$$\begin{aligned} B/\mathfrak{p} &\cong k[[X_1,\dots,X_d,Y]]/(X_1,\dots,X_l,Y,f) \\ &\cong k[[X_1,\dots,X_d,Y]]/(X_1,\dots,X_l,Y,a_r(0,\dots,0,X_{l+1},\dots,X_d)) \\ &\cong k[[X_{l+1},\dots,X_d]]/(a_r(0,\dots,0,X_{l+1},\dots,X_d)). \end{aligned}$$

On the other hand, since B is formally equidimensional, we have $\dim B/\mathfrak{p} = \dim B - \operatorname{ht}\mathfrak{p} = d-l$, and hence, by the above isomorphism and a dimension comparison, we have $a_r(0,\dots,0,X_{l+1},\dots,X_d) = 0$. This also shows that $f \in (X_1,\dots,X_l,Y)T$, proving (a), (b) and (c).

For part (d), if $\mathfrak{p}^{(n)} = \mathfrak{p}^n B_{\mathfrak{p}} \cap B$ denotes the n-th symbolic power of $\mathfrak{p}$, for each $n \geq 1$ we need to prove that $\mathfrak{p}^{(n)} = \mathfrak{p}^n$. By using the S_2-ification of a Rees algebra, this can be obtained as a consequence of [7, Lemma 3.10]. For the sake of completeness and of avoiding the introduction of the concepts needed to state and prove [7, Lemma 3.10], we present here a self-contained argument. Assume that $\mathfrak{p}^{(n)} \nsubseteq \mathfrak{p}^n$ and let $\mathfrak{q} \in \operatorname{Spec}(B)$ with $\mathfrak{q} \supseteq \mathfrak{p}$ such that $\mathfrak{p}^{(n)}B_{\mathfrak{q}} \nsubseteq \mathfrak{p}^n B_{\mathfrak{q}}$. Moreover, choose $\mathfrak{q}$ minimal with this property. Note that we must have $\mathfrak{q} \neq \mathfrak{p}$. To simplify the notation, set $C = B_{\mathfrak{q}}$, which is a local ring with maximal ideal $\mathfrak{m} := \mathfrak{q}C$. Since $\mathfrak{p}$ is equimultiple, $\mathfrak{p}C$ is also an equimultiple prime ideal of C. Moreover $l = \operatorname{ht}\mathfrak{p} = \ell(\mathfrak{p}C) < \operatorname{ht}\mathfrak{q} = \dim C$. By the minimality of $\mathfrak{q}$, the C-module $\mathfrak{p}^{(n)}C/\mathfrak{p}^nC$ has finite length, so there exists $s \geq 1$ such that $\mathfrak{m}^s\mathfrak{p}^{(n)}C \subseteq \mathfrak{p}^nC$. Now let $z \in \mathfrak{p}^{(n)}C \setminus \mathfrak{p}^nC$ and consider $\mathcal{R} = \bigoplus_{k\in\mathbb{Z}}(\mathfrak{p}^kC)t^k \subseteq C[t,t^{-1}]$, the (extended) Rees algebra of $\mathfrak{p}C$. With this notation we have $zt^n \in C[t,t^{-1}]$ and $(\mathfrak{m}^s, t^{-n})zt^n \subseteq \mathcal{R}$.

We also have

$$\dim\mathcal{R} - 1 = \dim C > \ell(\mathfrak{p}C) = \dim \mathcal{R}/(\mathfrak{m},t^{-1})\mathcal{R},$$

so $\operatorname{ht}(\mathfrak{m},t^{-1})\mathcal{R} \geq 2$. On the other hand, t^{-1} is a non-zero-divisor on $\mathcal{R}$ and $\mathcal{R}/t^{-1}\mathcal{R}$ has no embedded prime ideals. Indeed, $\mathcal{R}/t^{-1}\mathcal{R} \cong \bigoplus_{k\geq 0}\mathfrak{p}^kC/\mathfrak{p}^{k+1}C$ is a localization of the associated graded ring $\mathcal{G}_{\mathfrak{p}}(B) = \bigoplus_{k\geq 0}\mathfrak{p}^k/\mathfrak{p}^{k+1}$. Also, as $B = T/(f)$, $\mathcal{G}_{\mathfrak{p}}(B)$ is isomorphic to $\mathcal{G}_P(T)/(f^*)$, where f^* is the initial form of f in $\mathcal{G}_P(T) = \bigoplus_{k\geq 0}P^k/P^{k+1}$, which is a polynomial ring over $T/P \cong k[[X_{l+1},\dots,X_d]]$. Hence $\mathcal{R}/t^{-1}\mathcal{R}$ has no embedded prime ideals, which implies that the ideal $(\mathfrak{m},t^{-1})\mathcal{R}$ contains a regular sequence in $\mathcal{R}$ of length two. As $(\mathfrak{m}^s, t^{-n})\mathcal{R}$ and $(\mathfrak{m}, t^{-1})\mathcal{R}$ have the same radical, the ideal $(\mathfrak{m}^s, t^{-n})\mathcal{R}$ also contains a regular sequence in $\mathcal{R}$ of length two. From this and $(\mathfrak{m}^s, t^{-n})zt^n \subseteq \mathcal{R}$ it follows that $zt^n \in \mathcal{R}$, contradicting $z \notin \mathfrak{p}^n C$. This finishes the proof that $\mathfrak{p}^n = \mathfrak{p}^{(n)}$ for all $n \geq 1$.

To prove part (e) we proceed by induction on d ($d \geq l+1$). If $d = l+1$, by (c) we know that $B/\mathfrak{p} \cong k[[X_{l+1}]]$, so $x_{l+1} \notin \mathfrak{p}$. But $\mathfrak{p}$ is the only associated prime of $B/\mathfrak{p}^n$ (by (d)), so x_{l+1} is a non-zero-divisor on $B/\mathfrak{p}^n$. Suppose that part (e) is true for d variables; we want to prove it for $B = k[[X_1, \ldots, X_{d+1}, Y]]/(f)$ where $f = Y^r + a_1(X_1, \ldots, X_{d+1})Y^{r-1} + \cdots + a_r(X_1, \ldots, X_{d+1})$. As above, $B/\mathfrak{p} \cong k[[X_{l+1}, \ldots, X_{d+1}]]$, so x_{l+1} is a non-zero-divisor on $B/\mathfrak{p}^n$.

Let

$$\begin{aligned} B' &:= B/x_{l+1}B \\ &\cong k[[X_1, \ldots, X_l, X_{l+2}, \ldots, X_{d+1}, Y]]/(f(X_1, \ldots, X_l, 0, X_{l+2}, \ldots, X_{d+1})). \end{aligned}$$

First we prove that $\mathfrak{p}' := \mathfrak{p}B' = (x_1, \ldots, x_l, y)B'$ is an equimultiple ideal of B' of height l. Note that since $\ell(\mathfrak{p}) = l$ we must have $\ell(\mathfrak{p}') \leq l$. We also have

$$B'/\mathfrak{p}' \cong B/(x_1, \ldots, x_l, x_{l+1}, y) \cong \frac{(B/\mathfrak{p})}{x_{l+1}(B/\mathfrak{p})},$$

which implies that $\dim B'/\mathfrak{p}' = \dim B/\mathfrak{p} - 1 = \dim B - \operatorname{ht} \mathfrak{p} - 1 = d - l$. Then, since B' is formally equidimensional, $\operatorname{ht} \mathfrak{p}' = \dim B' - \dim B'/\mathfrak{p}' = d - (d - l) = l$. Therefore we have $l = \operatorname{ht} \mathfrak{p}' \leq \ell(\mathfrak{p}') \leq l$, showing that $\mathfrak{p}'$ is an equimultiple ideal of height l. By the induction hypothesis we obtain that $x_{l+2}, \ldots, x_{d+1}$ is a regular sequence on $B'/(\mathfrak{p}')^n$ for all $n \geq 1$. However,

$$B'/(\mathfrak{p}')^n \cong \frac{B/\mathfrak{p}^n}{x_{l+1}(B/\mathfrak{p}^n)}$$

and x_{l+1} is a non-zero-divisor on $B/\mathfrak{p}^n$. Therefore $x_{l+1}, x_{l+2}, \ldots, x_{d+1}$ is a regular sequence on $B/\mathfrak{p}^n$ for all n.

Part (f) is a direct consequence of (e) and the following well known general statement: If I is an ideal in a noetherian ring R and $a_1, \ldots, a_t \in R$ is a regular sequence on R/I, then $(a_1, \ldots, a_t) \cap I = (a_1, \ldots, a_t)I$. Indeed, if $J = (a_1, \ldots, a_t)$, then $(I \cap J)/IJ = \operatorname{Tor}_1^R(R/I, R/J) = 0$ (see [5, 1.1.4]). $\square$

We are now ready to prove the generalization of Hickel's result [12, Theorem 1.1] for equimultiple ideals.

Theorem 3.2. *Let $(R, \mathfrak{m})$ be an unmixed local ring containing a field and J an equimultiple ideal of R. Let $q = q(J) = \dim R/J$. If $y \in R$ is an element integral over J, then y satisfies an equation of integral dependence $y^r + a_1 y^{r-1} + \cdots + a_r = 0$ with $a_i \in J^i$ and $r \leq c_q(J)$.*

Proof. Note that the conclusion is equivalent to proving that $J(J + yR)^{r-1} = (J + yR)^r$ for some $r \leq c_q(J)$. By Hickel's work, the theorem is proved when J is $\mathfrak{m}$-primary, therefore we may assume that J is not $\mathfrak{m}$-primary.

By following a procedure that is standard in multiplicity theory (see also [13, Discussion 11.3.3], [12, 1.1]), we reduce the problem to the case when R is a complete domain with infinite residue field. We begin by noting that we may assume that the residue field $R/\mathfrak{m}$ is infinite. This is accomplished, if need be, by passing to the faithfully flat extension $R' = R[X]_{\mathfrak{m}[X]}$. If $J' = JR'$, then $q(J) = q(J')$ and $c_q(J) = c_q(J')$. Moreover, $J(J+yR)^{r-1} = (J + yR)^r$ if and only if $J'(J' + yR')^{r-1} = (J' + yR')^r$.

We may also assume that R is complete and unmixed. This is done by passing to $\widehat{R}$, a faithfully flat extension of R. Further, we may also assume that R is an integral domain. Indeed, by (2.5.2) we know that

$$c_q(J) = \sum_{\mathfrak{p} \in \mathrm{Min}(R)} c_q(J(R/\mathfrak{p}))\, \lambda(R_\mathfrak{p}).$$

For each $\mathfrak{p} \in \mathrm{Min}(R)$, if y satisfies an equation of integral dependence over $J(R/\mathfrak{p})$ of degree $r_\mathfrak{p} = c_q(J(R/\mathfrak{p}))$, then there exist $a_{i,\mathfrak{p}} \in J^i$ such that

$$f_\mathfrak{p} := y^{r_\mathfrak{p}} + a_{1,\mathfrak{p}} y^{r-1} + \cdots + a_{r_\mathfrak{p},\mathfrak{p}} \in \mathfrak{p}.$$

Let $f := \prod_{\mathfrak{p} \in \mathrm{Min}(R)} f_\mathfrak{p}^{\lambda(R_\mathfrak{p})}$ and note that

$$f \in \prod_{\mathfrak{p} \in \mathrm{Min}(R)} p^{\lambda(R_\mathfrak{p})} \subseteq \bigcap_{\mathfrak{p} \in \mathrm{Min}(R)} p^{\lambda(R_\mathfrak{p})} = (0),$$

where the last equality follows by localizing at each $\mathfrak{p} \in \mathrm{Ass}(R) = \mathrm{Min}(R)$. The equality $f = 0$ shows that y satisfies an equation of integral dependence over J of degree $r = c_q(J)$.

Let $K := (x_1, \dots, x_l)$ be a minimal reduction of J, where $l = \ell(J) = \operatorname{ht} J$. Then K is an equimultiple ideal of height l and $c_q(K) = c_q(I)$ (see 2.5). Since any equation of integral dependence of y over K is also an equation of integral dependence over J, without loss of generality we may assume that $J = (x_1, \dots, x_l)$. Keeping in mind that $\dim R/J = d - l$, let $x_{l+1}, \dots, x_d \in R$ be such that $(x_{l+1}, \dots, x_d)(R/J)$ is a minimal reduction of the maximal ideal $\mathfrak{m}/J$ of R/J. In particular, this implies that $x_1, \dots, x_d$ is a system of parameters of R.

The next step is to reduce the problem to the hypersurface case by constructing a formal power series ring A over a coefficient field of R such that $A \subseteq R$ is finite. This is, again, more or less standard. It is the technique introduced by Scheja and also employed by Hickel. However, since our ideal J is not $\mathfrak{m}$-primary, we need to do it a bit more carefully by choosing a special set of parameters as above. More precisely, since R is a complete domain that contains a field, the ring R has a coefficient field k. Define $\phi : S = k[[X_1, \dots, X_d]] \to R$ by $\phi(f) = f(x_1, \dots, x_d)$. Considering R as an S-module, since $R/(x_1, \dots, x_d)R$ is a finitely generated module over $S/(X_1, \dots, X_d) = k$, by complete Nakayama's lemma [14, Theorem 8.4] it follows that R is a finitely generated S-module. Moreover, $S/\operatorname{Ker}\phi \subseteq R$ is a finite extension, hence $\dim S/\operatorname{Ker}\phi = \dim R = d$, which implies that $\operatorname{Ker}\phi = 0$, and hence ϕ is injective. Therefore have a finite extension $A := k[[x_1, \dots, x_d]] \subseteq R$, where A is an isomorphic copy of the formal power series ring $S = k[[X_1, \dots, X_d]]$.

Let $y \in \overline{J}$ and consider the finite extensions

$$A = k[[x_1, \dots, x_l, x_{l+1}, \dots, x_d]] \subseteq B := A[y] \subseteq R.$$

Since y is integral over the normal domain A, the ring $B = A[y] = A[[y]]$ is isomorphic to $A[Y]/P$, where P is a prime ideal containing all the equations of integral dependence of y over A, each of them of the form $y^s + a_1(x_1, \dots, x_d)y^{s-1} + \cdots + a_s(x_1, \dots, x_d) = 0$ with $a_i(x_1, \dots, x_d) \in A$. Note that $A[Y]/P = A[[Y]]/PA[[Y]]$ and since $PA[[Y]]$ is a non-zero prime ideal of height one in the unique factorization domain $T := A[[Y]]$, it follows that $PA[[Y]] = (f)$ is a principal ideal with $f = Y^r + a_1(x_1, \dots, x_d)Y^{r-1} + \cdots + a_r(x_1, \dots, x_d)$ for some $r \geq 1$ and $a_i(x_1, \dots, x_d) \in A$.

Since y is integral over the ideal $(x_1, \dots, x_l)R$ and $B \subseteq R$ is a finite extension, y is also integral over $(x_1, \dots, x_l)B$ ([13, 1.6.1]). In particular, y is integral over $(x_1, \dots, x_d)B$, so

$e_B((x_1,\dots,x_d),B) = e_B((x_1,\dots,x_d,y),B)$. Using (2.4) and the fact that A and B have the same residue field, we obtain

$$e_A((x_1,\dots,x_d),A)\cdot \operatorname{rank}_A B = e_B((x_1,\dots,x_d),B) = e_B((x_1,\dots,x_d,y),B). \tag{3.2.1}$$

On the other hand, $B \cong T/(f)$, so $e_B((x_1,\dots,x_d,y),B) = \operatorname{ord}_T(f)$, with $\operatorname{ord}_T(f) = \max\{n \mid f \in (X_1,\dots,X_d,Y)^n T\}$. Since $e_A((x_1,\dots,x_d),A) = 1$ and $\operatorname{rank}_A B = r$, it follows that $r = \operatorname{ord}_T(f)$. This equality shows that we must have $a_i(x_1,\dots,x_d) \in (x_1,\dots,x_d)^i A$ for all $i = 1,\dots,r$. If we set $I := (x_1,\dots,x_l,y)B$, the equality $f(y) = 0$ shows, in particular, that

$$I^r \subseteq I^{r-1}(x_1,\dots,x_l)B + (x_{l+1},\dots,x_d)B. \tag{3.2.2}$$

At this point we note that I is an equimultiple ideal of B of height l. This follows from the fact that IR is equimultiple and $B \subseteq R$ is a finite extension of formally equidimensional local rings. Indeed, let $Q_B \supseteq I$ be a prime ideal of B with $\operatorname{ht} Q_B = \operatorname{ht} I$. Let Q_R be a prime ideal of R such that $Q_R \cap B = Q_B$. Then $\operatorname{ht} I = \operatorname{ht} Q_B = \dim B - \dim B/Q_B = \dim R - \dim R/Q_R = \operatorname{ht} Q_R \geq \operatorname{ht} IR = l$. On the other hand, since y is integral over $(x_1,\dots,x_l)B$ we have $\ell(I) \leq l$. As $\ell(I) \geq \operatorname{ht} I$ we must have $l = \ell(I) = \operatorname{ht} I$. We can now apply Lemma 3.1 for $\mathfrak{p} = I$ and conclude that $I^r \cap (x_{l+1},\dots,x_d)B = I^r(x_{l+1},\dots,x_d)B$. On the other hand, using (3.2.2) we have

$$\begin{aligned} I^r &\subseteq I^{r-1}(x_1,\dots,x_l)B + I^r \cap (x_{l+1},\dots,x_d)B \\ &= I^{r-1}(x_1,\dots,x_l)B + I^r(x_{l+1},\dots,x_d)B \\ &\subseteq I^{r-1}(x_1,\dots,x_l)B + \mathfrak{m}_B I^r \end{aligned}$$

where $\mathfrak{m}_B$ is the maximal ideal of B. This shows that $I^r = I^{r-1}(x_1,\dots,x_l)B$, i.e. y satisfies an equation of integral dependence of degree r over the ideal $(x_1,\dots,x_l)B$, and hence over $J = (x_1,\dots,x_l)R$.

It remains to prove that $r \leq c_q(J)$. First observe that since R and B have the same residue field, we have $e_B((x_1,\dots,x_d),B)\cdot \operatorname{rank}_B R = e_R((x_1,\dots,x_d),R)$. As $r = e_B((x_1,\dots,x_d),B)$ (see 3.2.1), it follows that $r \leq e_R((x_1,\dots,x_d),R)$.

We next prove that $e_R((x_1,\dots,x_d),R) = c_q(J)$, which will finish the proof. Considering the system of parameters $x_1,\dots,x_d$ of R, by applying (2.3) and keeping in mind that

$R/(x_{l+1},\dots,x_d)$ is equidimensional ([13, B.4.4]), we obtain

$$\mathrm{e}_R((x_1,\dots,x_d),R)=\sum_{\mathfrak{p}\in\Lambda}\mathrm{e}_R((x_{l+1},\dots,x_d),R/\mathfrak{p})\,\mathrm{e}_{R_\mathfrak{p}}((x_1,\dots,x_l)R_\mathfrak{p}),$$

where Λ consists of all the minimal prime ideals $\mathfrak{p}$ over $J=(x_1,\dots,x_l)$. However, recall that for our chosen system of parameters, the elements $x_{l+1},\dots,x_d$ generate a (minimal) reduction of the maximal ideal of $R/(x_1,\dots,x_l)$, so

$$\mathrm{e}_R((x_{l+1},\dots,x_d),R/\mathfrak{p})=\mathrm{e}_R(\mathfrak{m},R/\mathfrak{p})=\mathrm{e}_R(R/\mathfrak{p}) \text{ for each } \mathfrak{p}\in\Lambda.$$

On the other hand, from (2.5.1)

$$c_q(J)=\sum_{\mathfrak{p}\in\Lambda}\mathrm{e}_R(R/\mathfrak{p})\,\mathrm{e}_{R_\mathfrak{p}}((x_1,\dots,x_l)R_\mathfrak{p}),$$

which implies that $c_q(J)=\mathrm{e}_R((x_1,\dots,x_d),R)$, finishing the proof. □

Remark 3.3. In the case of an $\mathfrak{m}$-primary ideal J, each element of $\overline{J}$ satisfies an equation of integral dependence over J of degree $\mathrm{e}_A(J)$. At the same time, as proved by Rees, the Hilbert-Samuel multiplicity $\mathrm{e}_A(J)$ gives a numerical characterization of the integral closure of J. From this point of view, when J is just equimultiple, it is perhaps not surprising that $c_q(J)$ gives an upper bound for the degree of an equation of integral dependence because, as noted in (2.6), $c_q(J)$ characterizes the integral closure of equimultiple ideals the same way the classical Hilbert-Samuel multiplicity does it for $\mathfrak{m}$-primary ideals.

If the characteristic of the field contained in R is zero, the following corollary shows that the big reduction number of an equimultiple ideal is at most $c_q(J)-1$.

Corollary 3.4. *Let $(R,\mathfrak{m})$ be an unmixed local ring containing a field of characteristic zero and J an equimultiple ideal of R. If K is a minimal reduction of J, then $r_K(J)\le c_q(J)-1$.*

Proof. We first observe that for any ideal I in a noetherian ring R that contains a field of characteristic zero and each $n\ge 1$, the ideal I^n is generated by $\{x^n \mid x\in I\}$. In order to see this, choose a finite set of generators for $I=(a_1,\dots,a_t)$. In the polynomial ring $R[X_1,\dots,X_n]$ we have the identity

$$n!X_1\cdots X_n=\sum(-1)^{n-s}(X_{i_1}+\cdots+X_{i_s})^n,$$

where the sum runs over all $s \in \{1, \dots, n\}$ and all $(i_1, \dots, i_s)$ with $1 \le i_1 < \ldots < i_s \le n$. Since $n!$ is invertible in R, this implies that $a_1^{\alpha_1} \cdots a_t^{\alpha_t} \in \langle x^n \mid x \in I\rangle$ for every $\alpha_1, \dots, \alpha_t \ge 0$ with $\alpha_1 + \cdots + \alpha_t = n$, and hence $I^n = \langle x^n \mid x \in I\rangle$.

Set $r := c_q(J)$ and let $x \in J$ arbitrary. Since the ideal K is equimultiple, by Theorem 3.2 the element x satisfies an equation of integral dependence over K of degree at most $r = c_q(K) = c_q(J)$. It follows that $x^r \in K(K + xR)^{r-1} \subseteq KJ^{r-1}$ for every $x \in J$ and thus $J^r = KJ^{r-1}$. $\square$

Remark 3.5. Under the assumptions of the previous corollary, if J is an $\mathfrak{m}$-primary ideal and K is a minimal reduction of J, then $c_q(J) = \mathrm{e}_R(J)$ and $r_K(J) \le \mathrm{e}_R(J) - 1$. Since the ideal J is $\mathfrak{m}$-primary, Theorem 3.2 is not needed. The inequality follows directly from Hickel's theorem and the argument used in Corollary 3.4.

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