DERIVATIONS AND ROOT CLOSURES OF GRADED IDEALS

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Abstract. We show that the graded structure of an ideal is preserved under root closures. As an application, for a locally nilpotent derivation δ on a noetherian integral domain R containing a field of characteristic zero, we show that if I is δ-invariant, then so are its root closures.

1. Introduction

In this note we further explore properties of several root closures of ideals introduced in [8], primarily their behaviour in graded commutative noetherian rings. If I is an ideal in a commutative ring, the ideals $I^{\#}$, $I^{\boxed{\#}}$ and $I^{\natural}$ are constructed in [8] by capturing, with an increasing degree of refinement, n-th roots of elements in I^n for all $n \geq 1$. While technically only $I^{\boxed{\#}}$ and $I^{\natural}$ are closure operations, we collectively refer to all these constructions as root closures of I.

If I is a graded ideal of a graded commutative noetherian ring $R = \oplus_{i \geq 0} R_i$ and R_0 has infinite residue fields, we show that the root closures of I are graded, too (2.3, 2.4). We also note that the property of these closures of preserving gradings is connected with the property of invariance under locally nilpotent derivations (3.5). More precisely, if $\delta : R \to R$ is a locally nilpotent derivation on a noetherian integral domain R containing a field of characteristic zero and I is δ-invariant, i.e. $\delta(I) \subseteq I$, we show that $\delta(I^{\#}) \subseteq I^{\#}$, $\delta(I^{\boxed{\#}}) \subseteq I^{\boxed{\#}}$, and $\delta(I^{\natural}) \subseteq I^{\natural}$ (3.7). Along the way, we leave several unanswered questions regarding the behaviour of these closures when passing to a faithfully flat extension (2.4) or the root closure of the ring (3.12), and a certain (desirable) property in a formal power series ring (3.8).

We begin with a description of the concept of root closure in the context of rings followed by the construction of the root closures of an ideal.

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1.1 (Root closure of rings)**.** If $A \subseteq B$ is an extension of commutative rings, the ring A is said to root closed in B if A contains all the elements $x \in B$ such that $x^n \in A$ for some $n \geq 1$. The root closure $\mathcal{R}(A, B)$ of A in B is the smallest subring of B containing A that is root closed in B. In this generality, these concepts were introduced in [5], although the property of an integral domain of being root closed in its quotient field appeared much earlier in work of Sheldon [13] and later developed by Brewer, Costa, and McCrimmon [6]. The study of the root closure and its connections with the closely related concept of seminormality of integral domains was further expanded by, among others, D.F. Anderson [1,2]. We refer the reader to the survey papers [3] and [11] for detailed accounts of these concepts and their chronological development.

As described in [4], the root closure $\mathcal{R}(A, B)$ can be obtained recursively as follows. Let $\mathcal{R}_1(A, B)$ be the A-subalgebra of B generated by all the elements $b \in B$ such that $b^n \in A$ for some $n \geq 1$. Note that each element $x \in \mathcal{R}_1(A, B)$ is a finite sum $x = b_1 + b_2 + \cdots + b_r$ of elements $b_i \in B$ such that $b_i^n \in A$ for some n $(i = 1, \ldots, r)$. For $\ell \geq 2$ define $\mathcal{R}_\ell(A, B) = \mathcal{R}_1(\mathcal{R}_{\ell-1}(A, B))$. Then $\mathcal{R}(A, B) = \cup_{\ell \geq 1} \mathcal{R}_\ell(A, B)$ ([4, Proposition 1.1]).

1.2 (Root closures of ideals)**.** In the case of ideals in a commutative ring, several closely related concepts have been introduced in [8]. An ideal I in a commutative ring R is said to be root closed if I contains all the elements $a \in R$ such that $a^n \in I^n$ for some $n \geq 1$. For an arbitrary ideal $I \subseteq R$, let $I^\#$ be the ideal generated by all the elements a such that $a^n \in I^n$ for some $n \geq 1$. Each element $x \in I^\#$ is a finite sum $x = a_1 + \cdots + a_r$ with $a_i^n \in I^n$ for some $n \geq 1$ $(i = 1, \ldots, r)$. Set $I^{\#_1} = I^\#$ and define recursively $I^{\#_\ell} = (I^{\#_{\ell-1}})^\#$ for $\ell \geq 2$. Let $I^{\boxed{\#}} = \cup_{\ell \geq 1} I^{\#_\ell}$. Then $I^{\boxed{\#}}$ is the root closure of I, i.e., it is the smallest ideal containing I that is root closed ([8, Proposition 2.7]). We refer the reader to [8] for more elementary properties of this closure operation.

Remark 1.3. It is immediate from definition that $I^\#$ is contained in the integral closure $\overline{I}$ of I and since $\overline{I}$ is root closed, it follows that $I^{\boxed{\#}} \subseteq \overline{I}$. In fact, if I is a monomial ideal in a polynomial ring $k[X_1, \ldots, X_n]$ over a field k, then $\overline{I}$ is also monomial and a monomial $m \in \overline{I}$ if and only if $m^n \in I^n$ for some $n \geq 1$ ([14, Proposition 12.1.2]). Thus $I^\# = \overline{I}$ and $I^{\boxed{\#}} = \overline{I}$. For arbitrary ideals however, the inclusion $I^{\boxed{\#}} \subseteq \overline{I}$ may be strict as seen in several examples in [8, Section 4]. While all such examples in [8] are in positive characteristic, we fill this gap by providing one in characteristic zero later in Example 2.6.

1.4 (Root closures of Rees Algebras)**.** For an ideal I of R, consider the root closure $\mathcal{R}(R[It], R[t])$ of the Rees algebra $R[It]$ in $R[t]$. Since the root closure of a graded ring is graded ([11, 1.5]), it follows that $\mathcal{R}(R[It], R[t]) = \oplus_{i\geq 0} J_i t^i$ with J_i ideals in R ($J_0 = R$). A description of the graded components J_i is given in [8]. If we denote $I^{\natural} = \{a \in R \mid a^n \in (I^n)^{\boxplus} \text{ for some } n \geq 1\}$, then $I^{\natural}$ is an ideal ([8, 2.5]) and $J_i = (I^i)^{\natural}$ for all i ([8, 3.1]). It is worth noting that while $I^{\boxplus} \subseteq I^{\natural}$, the inclusion may be strict (see [8, Section 4]).

2. Root closures of graded ideals

In this section we look at root closures of graded ideals in graded commutative noetherian rings. We begin with the following consequence of the Chinese Remainder Theorem.

Lemma 2.1. *Let R be a commutative ring such that all its residue fields are infinite and let $I_1, I_2, \dots, I_t$ be a finite family of ideals in R. Then there exists a sequence $\{y_n\}_{n\geq 1}$ of elements in R such that for every $r = 1, \dots, t$*

$$\overline{y}_i \neq \overline{y}_j \text{ in } R/I_r \text{ for } i \neq j. \tag{2.1.1}$$

Proof. If all the ideals $I_1, I_2, \dots, I_t$ are contained in the same maximal ideal $\mathfrak{m}$, simply choose $\{y_n\}_{n\geq 1}$ such that all the elements $\{\overline{y}_n\}_{n\geq 1}$ in $R/\mathfrak{m}$ are distinct. The condition (2.1.1) is then satisfied. In the general case, let $\mathfrak{m}_1, \dots, \mathfrak{m}_s$ ($s \geq 2$) be a finite family of distinct maximal ideals such that every ideal I_j is contained in one of these maximal ideals. For each $k = 1, \dots, s$, choose a sequence of elements $\{x_{k,n}\}_{n\geq 1}$ such that the elements of the sequence $\{\overline{x}_{k,n}\}_{n\geq 1}$ are distinct in $R/\mathfrak{m}_k$.

Using the Chinese Remainder theorem, for each $n \geq 1$ choose $y_n \in R$ such that

$$\begin{aligned} y_n &\equiv x_{1,n} (\text{mod } \mathfrak{m}_1) \\ y_n &\equiv x_{2,n} (\text{mod } \mathfrak{m}_2) \\ &\vdots \\ y_n &\equiv x_{s,n} (\text{mod } \mathfrak{m}_s). \end{aligned}$$

Then, for every $k \in \{1, \dots, s\}$, in $R/\mathfrak{m}_k$ we have $\overline{y}_i = \overline{x}_{k,i} \neq \overline{x}_{k,j} = \overline{y}_j$ whenever $i \neq j$. The condition (2.1.1) is then satisfied. $\square$

The following lemma is well-known in the case when the graded ring R contains an infinite field. We state and prove a version that only requires that all the residue fields of R_0 be infinite.

Lemma 2.2. *Let $R = \oplus_{i\geq 0} R_i$ be a graded commutative noetherian ring such that all the residue fields of R_0 are infinite. For each $\alpha \in R_0$, let $T_\alpha : R \to R$ be the graded ring homomorphism defined by*

$$T_\alpha(x_0 + x_1 + \cdots + x_d) = x_0 + x_1\alpha + \cdots + x_d\alpha^d,$$

where $x_i \in R_i$ are homogeneous elements of R.

If I is an ideal of R, then I is graded if and only if $T_\alpha(I) \subseteq I$ for all $\alpha \in R_0$.

Proof. If I is graded, for $x \in I$ we can write $x = x_0 + x_1 + \cdots + x_d$ with $x_i \in I_i = I \cap R_i$, and it is clear that $T_\alpha(x) \in I$ for all $\alpha \in R_0$.

We now assume $T_\alpha(I) \subseteq I$ for all $\alpha \in R_0$. Since R is noetherian, there are finitely many associated prime ideals $Q_1, \ldots, Q_t$ of R/I. Let $\mathfrak{p}_r = Q_r \cap R_0$ for $r = 1, \ldots, t$. As in the previous lemma, choose a sequence $\{\alpha_n\}_{n\geq 1}$ of elements in R_0 such that $\overline{\alpha}_i \neq \overline{\alpha}_j$ in $R_0/\mathfrak{p}_r$ for all r and all $i \neq j$.

Let $x = x_0 + x_1 + \cdots + x_d \in I$ with $x_i \in R_i$. We want to prove that $x_i \in I$ for all i. Since $T_{\alpha_1}(x), T_{\alpha_2}(x), , \ldots, T_{\alpha_{d+1}}(x) \in I$, there exist $b_1, b_2, \ldots, b_{d+1} \in I$ such that

$$\begin{gathered} x_0 + x_1\alpha_1 + \cdots + x_d\alpha_1^d = b_1 \\ x_0 + x_1\alpha_2 + \cdots + x_d\alpha_2^d = b_2 \\ \vdots \\ x_0 + x_1\alpha_{d+1} + \cdots + x_d\alpha_{d+1}^d = b_{d+1} \end{gathered}$$

With the notation

$$A = \begin{bmatrix} 1 & \alpha_1 & \ldots & \alpha_1^d \\ 1 & \alpha_2 & \ldots & \alpha_2^d \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \alpha_{d+1} & \ldots & \alpha_{d+1}^d \end{bmatrix}, \text{ we have } A \cdot \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_d \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_d \end{bmatrix},$$

and after multiplying on both sides with the adjoint $\mathrm{adj}(A)$, it follows that $\det(A)x_i \in I$ for all $i = 0, \ldots, d$. But $\det(A) = \prod_{1\leq i<j\leq d+1}(\alpha_j - \alpha_i)$, and since $\alpha_j - \alpha_i \notin \mathfrak{p}_r$ for $i < j$, it follows that $\det(A) \notin \mathfrak{p}_r$ for all r. Then $\det(A) \notin \cup_{r=1}^{s} Q_r$, i.e., $\det(A)$ is a non-zero-divisor on R/I. From $\det(A)x_i \in I$ we then obtain $x_i \in I$ for all $i = 0, \ldots, d$, finishing the proof. □

Proposition 2.3. *Let $R = \oplus_{i\geq 0} R_i$ be a graded commutative noetherian ring such that all the residue fields of R_0 are infinite and let I be a graded ideal of R. Then $I^{\#}$, $I^{\boxed{\#}}$ and $I^{\natural}$ are graded.*

Proof. We first show that $I^{\#}$ is graded. By Lemma 2.2, it is enough to show that $T_\alpha(I^{\#}) \subseteq I^{\#}$ for all $\alpha \in R_0$. If $x \in R$ such that $x^n \in I^n$, then $T_\alpha(x)^n = T_\alpha(x^n) \in T_\alpha(I^n) \subseteq I^n$, so $T_\alpha(x) \in I^{\#}$. Since I is generated by elements x such that $x^n \in I^n$ for some $n \geq 1$, it follows that $T_\alpha(I^{\#}) \subseteq I^{\#}$, so $I^{\#}$ is graded. Recursively, we get that $I^{\boxed{\#}}$ is graded, too. To show that $I^{\natural}$ is graded, let $x = x_0 + \cdots + x_d \in I^{\natural}$ with $x_i \in R_i$ and $x_d \neq 0$. We will use induction on d to show that $x_i \in I^{\natural}$ for all i. This is clear for $d = 0$. For $d \geq 1$, since $x^n \in (I^n)^{\boxed{\#}}$ for some n and $(I^n)^{\boxed{\#}}$ is graded, it follows that $x_d^n \in (I^n)^{\boxed{\#}}$, so $x_d \in I^{\natural}$. Then $x - x_d = x_0 + \cdots + x_{d-1} \in I^{\natural}$, and the induction hypothesis shows that $x_i \in I^{\natural}$ for all i. □

Remark 2.4. A different way to prove that $I^{\natural}$ is graded is by using the fact that the root closure of the Rees algebra $R[It]$ in $R[t]$ is $\oplus_{n\geq 0}(I^n)^{\natural} t^n$ (1.4). As noted in [11, 1.5], if $A \subseteq B$ is a graded extension of rings, then the root closure of A in B is graded, too. An adaptation of this result for bigraded extensions applied to the bigraded extension $R[It] \subseteq R[t]$ will show that $I^{\natural}$ is graded (in the grading of the ring R). Note that this approach does not need that the residue fields of R_0 be infinite. However, in the case of $I^{\#}$ and $I^{\boxed{\#}}$, we are unable to find a proof that avoids this assumption on the residue fields. A natural path to follow would be to extend to a faithfully flat extension of R that has infinite residue fields and then contract back to R. To do so, it would be enough to know that if $R \to S$ is a faithfully flat extension, then $(IS)^{\#} \cap R = I^{\#}$ and $(IS)^{\boxed{\#}} \cap R = I^{\boxed{\#}}$. At this time, we are unable to settle whether or not this is true in general.

We conclude with an example of a graded ideal with $I^{\#} \subsetneqq (I^{\#})^{\#}$. In particular $I^{\#} \subsetneqq \overline{I}$. The example is relevant not only because the graded structure of the ring is essential in establishing the proper containment, but also because the ring is an integral domain containing a field of characteristic zero, a set-up that will be used later in Section 3. (All previous examples in [8, Section 4] were in rings of positive characteristic.)

We need the following lemma already proved in [8].

Lemma 2.5. ([8, Lemma 4.2]) *Let $R = \oplus_{i\geq 0} R_i$ be a graded commutative ring where $R_0 = k$ is a field. Let $I \subseteq R$ be a graded proper ideal of R, and let $x \in I^{\#}$ be a non-zero homogeneous element of degree 1. Then there exist $a_1, \dots, a_s$ homogeneous elements of degree 1 and $n \geq 1$ such that $x = a_1 + \cdots + a_s$ with $a_i^n \in I^n$ for all i.*

Example 2.6. Let $R = \dfrac{\mathbb{Q}[X,Y,Z]}{\langle X^2+Y^2, Z^2-X^2-XY\rangle}$, where X, Y, and Z are indeterminates of degree 1. Denote by x, y, and z the residue classes in R of X, Y, and Z, respectively, and let $I = \langle x\rangle \subseteq R$. We will show $I^{\#}$ is strictly contained in $(I^{\#})^{\#}$ by proving $z \in (I^{\#})^{\#} \setminus I^{\#}$. Moreover, using Gröbner bases methods as implemented in [9, Macaulay2], one can show that $\langle X^2+Y^2, Z^2-X^2-XY\rangle$ is a prime ideal in $\mathbb{Q}[X,Y,Z]$, so R is an integral domain.

First, notice that $y^2 = -x^2 \in I^2$, so $y \in I^{\#}$. Also, $z^2 = x^2 + xy \in (I^{\#})^2$, so $z \in (I^{\#})^{\#}$. To prove $z \notin I^{\#}$, we proceed by contradiction. Assume $z \in I^{\#}$. By Lemma 2.5, we may write $z = a_1 + \cdots + a_s$ with each a_i homogeneous of degree 1 and $a_i^n \in I^n$ for some $n \geq 1$. There exists i such that $a_i = c_1x + c_2y + c_3z$ for some $c_1, c_2, c_3 \in \mathbb{Q}$ with $c_3 \neq 0$. Indeed, supposing otherwise, we deduce $z \in \langle x, y\rangle$. Lifting in $\mathbb{Q}[X,Y,Z]$, we find $Z \in \langle X, Y, X^2+Y^2, Z^2-X^2-XY\rangle = \langle X, Y, Z^2\rangle$, which is not true. To simplify the notation, set $a = c_1x + c_2y + c_3z$ ($c_1, c_2, c_3 \in \mathbb{Q}$, $c_3 \neq 0$). We also have $a^n \in I^n$ for some $n \geq 1$. First, observe that $n > 1$. Indeed, otherwise we deduce again $z \in \langle x, y\rangle$, which is not true. Without loss of generality, we may assume $c_3 = 1$. Since $a^n \in I^n$, we obtain

$$(c_1X + c_2Y + Z)^n \in \langle X^n, X^2+Y^2, Z^2-X^2-XY\rangle. \tag{2.6.1}$$

Write

$$(c_1X + c_2Y + Z)^n = p_1X^n + p_2(X^2+Y^2) + p_3(Z^2-X^2-XY) \tag{2.6.2}$$

where $p_1 \in \mathbb{Q}$, and $p_2, p_3 \in \mathbb{Q}[X,Y,Z]$ are homogeneous polynomials of degree $(n-2)$.

By setting $X = 0$ and $Y = 0$ in (2.6.2), it follows that that the coefficient of Z^{n-2} in the polynomial p_3 must be 1. This implies the expansion of the expression $p_3(Z^2-X^2-XY)$ contains the term $-XYZ^{n-2}$. However, the expansion of $p_1X^n+p_2(X^2+Y^2)$ does not contain the monomial XYZ^{n-2} (it has coefficient 0), from which we deduce that the expansion of the left-hand side of (2.6.2) contains the term $-XYZ^{n-2}$. On the other hand, it can be seen from a simple counting argument that the coefficient of XYZ^{n-2} on the left-hand side of (2.6.2) is $n(n-1)c_1c_2$. Hence we obtain

$$n(n-1)c_1c_2 = -1.$$

Since $a^n \in I^n$ implies $a^{2n} \in I^{2n}$, we deduce that the equation (2.6.1) is still valid if $2n$ is substituted in place of n. Repeating the earlier argument using $2n$ in place of n, we deduce that

$$2n(2n-1)c_1c_2 = -1.$$

Then $(n-1) = 2(2n-1)$, which is not possible.

3. Derivations and Root Closure of rings and Ideals

Let R be a commutative noetherian ring and $\delta : R \to R$ be a derivation on R. We say that an ideal I of R is δ-invariant if $\delta(I) \subseteq I$, where $\delta(I)$ is the ideal generated by all the elements $\delta(f)$ with $f \in I$. In [7], it is proved that the integral closure, coefficient ideals, and rational powers of an δ-invariant ideal are δ-invariant, too. The main goal of this section is to prove similar properties for the root closures $I^{\#}, I^{\boxed{\#}}$ and $I^{\natural}$. We also establish a connection between these properties and the property of preserving the grading when passing from a graded ideal to one of its root closures.

A well-known result of Seidenberg [12, Section 3] shows that if δ is a derivation on the quotient field of a noetherian integral domain R containing a field of characteristic zero and $\delta(R) \subseteq R$, then $\delta(\overline{R}) \subseteq \overline{R}$, where $\overline{R}$ is the integral closure of R in its quotient field. We will show that a similar result holds for the root closure $R^{\boxed{\#}} = \mathcal{R}(R, Q(R))$ of R in its quotient field.

Remark 3.1. If R is a noetherian integral domain (more generally, a Mori domain), then R is root closed (in its quotient field) if and only if $R[[X]]$ is root closed (in its quotient field). We refer the reader to [5, Theorem 2.13] for a proof of this result and a detailed account of when a power series ring is root closed. In particular, if R is an integral domain whose root closure $R^{\boxed{\#}}$ (in its quotient field) is noetherian, then $R^{\boxed{\#}}[[X]]$ is root closed, and therefore $(R[[X]])^{\boxed{\#}} \subseteq R^{\boxed{\#}}[[X]]$. We also note that if R is noetherian and the integral closure $\overline{R}$ is module-finite over R, since $R \subseteq R^{\boxed{\#}} \subseteq \overline{R}$, $R^{\boxed{\#}}$ is module-finite over R, too; hence $R^{\boxed{\#}}$ is noetherian. In particular, if R is a local analytically unramified domain, then $R^{\boxed{\#}}$ is noetherian ([10, 9.2.2]).

Proposition 3.2. *Let R be an integral domain containing a field of characteristic zero. Assume that $R^{\boxed{\#}}$ is noetherian. Let $\delta : Q(R) \to Q(R)$ be a derivation such that $\delta(R) \subseteq R$. Then $\delta(R^{\boxed{\#}}) \subseteq R^{\boxed{\#}}$.*

Proof. The proof relies upon the same exponential function associated with the derivation δ that was used by Seidenberg in [12]. Let $E : R \to R[[t]]$ be given by

$$E = \sum_{i \geq 0} \frac{t^i}{i!}\delta^i = id + t\delta + \frac{t^2}{2!}\delta^2 + \cdots . \tag{3.2.1}$$

As noted in [12], E is an injective ring homomorphism which can be extended to $E : Q(R) \to Q(R[[t]])$ by

$$E(\alpha/\beta) = \frac{E(\alpha)}{E(\beta)} = \frac{\alpha}{\beta} + t\delta\left(\frac{\alpha}{\beta}\right) + \delta^2\left(\frac{\alpha}{\beta}\right) + \cdots \text{ for } \alpha, \beta \in R, \beta \neq 0.$$

We first show that $E(R^{\boxed{\#}}) \subseteq R^{\boxed{\#}}[[t]]$. To do this, we will show by induction that $E(R^{\#_\ell}) \subseteq R^{\boxed{\#}}[[t]]$ for $\ell \geq 1$, where $R^{\#_\ell} = \mathcal{R}_\ell(R, Q(R))$. For $\ell = 1$, it is enough to to prove that $E(\alpha) \in R^{\boxed{\#}}[[t]]$ for every $\alpha \in Q(R)$ with $\alpha^n \in R$ for some $n \geq 1$. In this case we have $E(\alpha)^n = E(\alpha^n) \in R[[t]]$, so $E(\alpha) \in (R[[t]])^\# \subseteq (R[[t]])^{\boxed{\#}} \subseteq R^{\boxed{\#}}[[t]]$, where the last inclusion follows from the results outlined in Remark 3.1. Assume $\ell \geq 2$ and let $\alpha \in Q(R)$ such that $\alpha^n \in R^{\#_{\ell-1}}$ for some $n \geq 1$. Then $E(\alpha)^n = E(\alpha^n) \in E(R^{\#_{\ell-1}}) \subseteq R^{\boxed{\#}}[[t]]$. But $R^{\boxed{\#}}[[t]]$ is root closed (Remark 3.1), so $E(\alpha) \in R^{\boxed{\#}}[[t]]$, finishing the inductive proof.

Since $E(R^{\boxed{\#}}) \subseteq R^{\boxed{\#}}[[t]]$, for every $\alpha \in R^{\boxed{\#}}$ we have

$$\alpha + t\delta(\alpha) + \frac{t^2}{2!}\delta^2(\alpha) + \cdots \in R^{\boxed{\#}}[[t]],$$

which implies that $\delta(\alpha) \in R^{\boxed{\#}}$. □

Before we continue, we make the following elementary observation.

Remark 3.3. For an arbitrary ideal in a commutative ring R, clearly $IR[[t]] \subseteq I[[t]]$ and $(I[[t]])^n \subseteq I^n[[t]]$. (The inclusions may be strict.) However, if I is a finitely generated ideal in R, then $IR[[t]] = I[[t]]$ and consequently $(I[[t]])^n = I^n[[t]]$.

Proposition 3.4. *The following are equivalent:*

(a) *For every derivation δ on a noetherian integral domain containing a field of characteristic zero and every ideal I, if $\delta(I) \subseteq I$, then $\delta(I^\#) \subseteq I^\#$.*
(b) *For every ideal I in a noetherian integral domain containing a field of characteristic zero $(I[[t]])^\# = I^\#[[t]]$.*

Proof. We begin by pointing out that the inclusion $(I[[t]])^\# \supseteq I^\#[[t]]$ always holds when the ring is noetherian. Indeed, since the ideal $I^\#$ is finitely generated, $I^\#[[t]] = I^\# R[[t]]$, so it is generated by elements $x \in R$ such that $x^n \in I^n$ for some $n \geq 1$. For each such element, $x^n \in I^n[[t]] = (I[[t]])^n$, so $x \in (I[[t]])^\#$.

$(a) \implies (b)$ We first show that if $f = \sum_{i\geq 0} f_i t^i \in (I[[t]])^\# \subseteq R[[t]]$ where $f_i \in R$, then $f_0 \in I^\#$. Indeed, if $f = g_1 + \cdots + g_r$ with all $g_i \in R[[t]]$ and $g_i^n \in I^n[[t]]$ for some $n \geq 1$, then $f_0 = f(0) = g_1(0) + \cdots + g_r(0)$ with $g_i(0)^n \in I^n$, hence $f_0 \in I^\#$. To show that all the other coefficients of f belong to $I^\#$, since (a) holds and considering the derivation $\frac{\partial}{\partial t}$ on $R[[t]]$, for every m we have

$$\frac{\partial^m f}{\partial t^m} = m! f_m + (\text{multiple of } t) \in (I[[t]])^\#.$$

As above, this implies that $f_m \in I^\#$. Therefore $(I[[t]])^\# \subseteq I^\#[[t]]$.

$(b) \implies (a)$ Let I be a δ-invariant ideal of R. As in the proof of Proposition 3.2, consider the ring homorphism $E = \sum_{i\geq 0} \frac{t^i}{i!}\delta^i : R \to R[[t]]$. Then $E(I) \subseteq I[[t]]$. We show that this implies that $E(I^\#) \subseteq I^\#[[t]]$. Indeed, if $\alpha \in R$ such that $\alpha^n \in I^n$ for some $n \geq 1$, then $E(\alpha)^n = E(\alpha^n) \in E(I^n) \subseteq I^n[[t]]$, which implies that $E(\alpha) \in (I[[t]])^\# = I^\#[[t]]$. Then, for every $\alpha \in I^\#$,

$$E(\alpha) = \alpha + t\delta(\alpha) + \frac{t^2}{2!}\delta^2(\alpha) + \cdots \in I^\#[[t]],$$

and therefore $\delta(\alpha) \in I^\#$. □

Recall that a derivation $\delta : R \to R$ is called locally nilpotent if for every $a \in R$ there exists $n \geq 1$ such that $\delta^n(a) = 0$. We then have the following.

Proposition 3.5. *The following (equivalent) statements are true:*

(a) *For every graded ideal I in a graded noetherian integral domain containing a field of characteristic zero, the ideal $I^\#$ is graded.*
(b) *For every locally nilpotent derivation δ on a noetherian integral domain containing a field of characteristic zero and every ideal I, if $\delta(I) \subseteq I$, then $\delta(I^\#) \subseteq I^\#$.*
(c) *For every ideal I in a noetherian integral domain containing a field of characteristic zero $(I[t])^\# = I^\#[t]$.*

Proof. If $R = \oplus_{i\geq 0} R_i$ is a graded commutative noetherian ring and all the residue fields of R_0 are infinite, we know from Proposition 2.3 that $I^\#$ is graded whenever I is graded. Thus (a) is true. We now prove the equivalence of (a), (b), and (c).

First, with the same argument used in the proof of Proposition 3.4, it should be noted that the inclusion $(I[t])^\# \supseteq I^\#[t]$ always holds.

The implication $(b) \implies (c)$ follows exactly the argument used in Proposition 3.4 for $(a) \implies (b)$ by using $R[t]$ instead of $R[[t]]$. (In this context, the derivation $\frac{\partial}{\partial t}$ is locally nilpotent.)

For the implication $(c) \implies (b)$, since δ is locally nilpotent, the ring homorphism E used to prove the implication $(b) \implies (a)$ in Proposition 3.4 has range in the polynomial ring $R[t]$ so it can be defined $E : R \to R[t]$. With this observation, using $I[t]$ instead of $I[[t]]$, the rest of the argument can be used to deduce $\delta(I^{\#}) \subseteq I^{\#}$.

We now prove $(a) \implies (c)$. By considering the graded ideal $I[t]$ in the t-graded ring $R[t]$, it follows that $(I[t])^{\#} = \oplus_{m\geq 0} J_m t^m$ is a graded ideal, too. To prove $J_m \subseteq I^{\#}$, let $a \in J_m$. Then there exist $g_1, \ldots, g_r \in R[t]$ with $g_i^n \in I^n[t]$ for some $n \geq 1$ $(i = 1, \ldots, r)$ such that $at^m = g_1 + g_2 + \cdots + g_r$. Then $a = g_1(1) + g_2(1) + \cdots + g_r(1)$, and since $g_i(1)^n \in I^n$ for all i we get $a \in I^{\#}$. Thus $(I[t])^{\#} \subseteq I^{\#}[t]$. As noted at the beginning, the opposite inclusion holds, too.

We now prove $(c) \implies (a)$. Let I be a graded ideal in $R = \oplus_{i\geq 0} R_i$ and let $f = f_0 + f_1 + \cdots + f_d \in I^{\#}$ with $f_i \in R_i$ homogeneous of degree i. Write $f = g_1 + \cdots + g_r$ with $g_i^n \in I^n$ for some $n \geq 1$ $(i = 1, \ldots, r)$. For each i, write $g_i = g_{i,0} + g_{i,1} + \cdots + g_{i,s}$ with $g_{i,j} \in R_j$ and let $G_i = g_{i,0} + g_{i,1}t + \cdots + g_{i,s}t^s \in R[t]$. Since $g_i^n \in I^n$ we have $G_i^n \in I^n[t]$ for all i. Let $F = G_1 + G_2 + \cdots + G_r$. Thus $F \in (I[t])^{\#} = I^{\#}[t]$. Note that $F = f_0 + f_1 t + \cdots + f_d t^d$ and this implies that $f_j \in I^{\#}$ for all j. □

Remark 3.6. By Proposition 2.3, statement (a) in the previous proposition is true for every ideal I in a graded commutative noetherian ring $R = \oplus_{i\geq 0} R_i$ such that R_0 has infinite residue fields. The argument in the proof of $(a) \implies (c)$ then shows that $(I[t])^{\#} = I^{\#}[t]$ for every ideal I in a commutative noetherian ring with infinite residue fields. Through iteration, we also obtain $(I[t])^{\boxed{\#}} = I^{\boxed{\#}}[t]$. Let us observe that $(I[t])^{\natural} = I^{\natural}[t]$ as well. By Proposition 2.3, $(I[t])^{\natural}$ is a graded ideal in $R[t]$ so we can write $(I[t])^{\natural} = \oplus_{i\geq 0} K_i t^i$ with K_i ideals in R. For $a \in K_i$ we have $(at^i)^n \in (I^n[t])^{\boxed{\#}}$ for some $n \geq 1$. As noted above, we have $(I^n[t])^{\boxed{\#}} = (I^n)^{\boxed{\#}}[t]$, so $a^n \in (I^n)^{\boxed{\#}}$, and therefore $a \in I^{\natural}$. This shows $(I[t])^{\natural} \subseteq I^{\natural}[t]$. For the opposite containment, if $a \in I^{\natural}$, then there exists n such that $a^n \in (I^n)^{\boxed{\#}} \subseteq (I^n)^{\boxed{\#}}[t] = (I^n[t])^{\boxed{\#}}$, so $a \in (I[t])^{\natural}$.

We also obtain the following.

Corollary 3.7. *Let δ be a locally nilpotent derivation on a noetherian integral domain R containing a field of characteristic zero. Let I be an ideal of R. If $\delta(I) \subseteq I$, then $\delta(I^{\#}) \subseteq I^{\#}$, $\delta(I^{\boxed{\#}}) \subseteq I^{\boxed{\#}}$, and $\delta(I^{\natural}) \subseteq I^{\natural}$.*

Proof. Part (b) of Proposition 3.5 already showed $\delta(I^{\#}) \subseteq I^{\#}$. Through iteration we also obtain $\delta(I^{\boxed{\#}}) \subseteq I^{\boxed{\#}}$.

Now let $\alpha \in I^{\natural}$ and consider again the ring homomorphism $E : R \to R[t]$ given by $E = \sum_{i \geq 0} \frac{t^i}{i!}\delta^i$. Since $\alpha^n \in (I^n)^{\boxed{\#}}$ for some $n \geq 1$ and $\delta((I^n)^{\boxed{\#}}) \subseteq (I^n)^{\boxed{\#}}$ we have

$$E(\alpha)^n = E(\alpha^n) = \sum_{i \geq 0} \frac{t^i}{i!}\delta^i(\alpha^n) \in (I^n)^{\boxed{\#}}[t] = (I^n[t])^{\boxed{\#}},$$

and therefore $E(\alpha) \in (I[t])^{\natural}$. However, as noted in Remark 3.6, $(I[t])^{\natural} = I^{\natural}[t]$, so all the coefficients of $E(\alpha)$ must be in $I^{\natural}$. This shows that $\delta(\alpha) \in I^{\natural}$. □

Remark 3.8. To obtain the same conclusions from Corollary 3.7 without assuming that the derivation δ is locally nilpotent, in view of Proposition 3.4, we will need to prove that $(I[[t]])^{\#} = I^{\#}[[t]]$. At this time, we are unable to decide whether or not this is true.

Remark 3.9. If R is root closed (in its field of fractions), we are able to prove that $\delta(I^{\natural}) \subseteq I^{\natural}$ without assuming that δ is locally nilpotent. We do this by applying Proposition 3.2 to the root closure of the Rees algebra $R[It]$. More precisely, we have the following.

Proposition 3.10. *Let R be a root closed integral domain containing a field of characteristic zero and let I be an ideal of R. Assume that the root closure $(R[It])^{\boxed{\#}}$ of $R[It]$ in $R[t]$ is noetherian. If $\delta : R \to R$ is a derivation and $\delta(I) \subseteq I$, then $\delta(I^{\natural}) \subseteq I^{\natural}$.*

Proof. We may assume that I is a non-zero ideal. Let T denote the root closure of the Rees algebra $R[It]$ in its quotient field $Q(R)(t)$, so that $(R[It])^{\boxed{\#}} \subseteq T$. We will show that in fact $(R[It])^{\boxed{\#}} = T$.

Since R is root closed in $Q(R)$, the ring $R[t]$ is root closed in $Q(R)(t)$ ([6, Theorem 2]), which implies that $T \subseteq R[t]$. We claim that $(R[It])^{\boxed{\#}}$ is root closed in $Q(R)(t)$, and therefore $T \subseteq (R[It])^{\boxed{\#}}$. Indeed, let $\alpha \in Q(R)(t)$ such that $\alpha^n \in (R[It])^{\boxed{\#}}$ for some $n \geq 1$. Then $\alpha^n \in T$, and since T is root closed in $Q(R)(t)$, we must have $\alpha \in T$. However, as noted above, $T \subseteq R[t]$, so $\alpha \in R[t]$, and since $(R[It])^{\boxed{\#}}$ is root closed in $R[t]$ we obtain $\alpha \in (R[It])^{\boxed{\#}}$.

With $\delta(\sum a_i t^i) = \sum \delta(a_i)t^i$ for $a_i \in R$, the derivation δ extends naturally to $\delta : R[t] \to R[t]$ and further to $\delta : Q(R[t]) \to Q(R[t])$. Since $\delta(I) \subseteq I$, we also have $\delta(R[It]) \subseteq R[It]$. By Proposition 3.2 we then obtain $\delta(T) \subseteq T$, so $\delta((R[It])^{\boxed{\#}}) \subseteq (R[It])^{\boxed{\#}}$. On the other hand, as proved in [8, Theorem 3.1], $(R[It])^{\boxed{\#}} = \oplus_{n\geq 0}(I^n)^{\natural}t^n$. For $a \in I^{\natural}$ we then have $\delta(at) = \delta(a)t \in I^{\natural}t$, hence $\delta(a) \in I^{\natural}$. □

Remark 3.11. The assumption $(R[It])^{\boxed{\#}}$ noetherian is a mild constraint. For instance, if R is a local analytically unramified domain, then that integral closure of the Rees algebra $R[It]$ in $R[t]$ is module-finite over $R[It]$ ([10, 9.2.1]). This implies that $(R[It])^{\boxed{\#}}$ is module-finite over $R[It]$, too, hence $(R[It])^{\boxed{\#}}$ is a noetherian ring.

Remark 3.12. To drop the assumption that R is root closed in Proposition 3.10, one may try to pass from R to its root closure $R^{\boxed{\#}}$. However, to contract to the initial ring R, one will need to know that $(IR^{\boxed{\#}})^{\boxed{\#}} \cap R = I^{\boxed{\#}}$. We do not know whether or not this equality is true in general.

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