CONTINUED FRACTION EXPANSION

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Math Club

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Outline:

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1. Motivation

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2. The Continued Fraction Expansion

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3. Fundamental Properties of Continued Fraction Expansion

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2. The Continued Fraction Expansion
3. Fundamental Properties of Continued Fraction Expansion
4. Approximation of Real Numbers with Rationals

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Remark. There's nothing special about 10 (!!); one can replace it by any positive integer $n \geq 2$, and the resulting expansion is called as the *n-ary expansion.*

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- Let a_0 be the integer part of x.
- Let $x_1 = x - a_0$ (fractional part of x); divide $[0, 1)$ into 10 equal sub-intervals (of length $\frac{1}{10}$), and if $x_1 \in [\frac{i}{10}, \frac{i+1}{10})$, then let $a_1 = i$. (Note: $i \in \{0, 1, 2, \ldots, 9\}$.)

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- Let $x_2 = x_1 - \frac{a_1}{10}$; divide $[\frac{i}{10}, \frac{i+1}{10})$ into 10 sub-intervals (of length $\frac{1}{10^2}$), and if $x_2 \in [\frac{i}{10} + \frac{j}{10^2}, \frac{i}{10} + \frac{j+1}{10^2})$, then let $a_2 = j$. (Note: $j \in \{0, 1, 2, \ldots, 9\}$.)

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Continuing in this manner, we get the decimal expansion of x as $x = a_0.a_1a_2a_3\ldots$, where $a_k \in \{0, 1, \ldots, 9\}$.

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$$a_0 + \frac{a_1}{10} + \cdots + \frac{a_n}{10^n} \leq x < a_0 + \frac{a_1}{10} + \cdots + \frac{a_n}{10^n} + \frac{1}{10^n}.$$

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3. Some rational numbers have more than one “different” decimal expansions.
4. This problem is remedied by the map $T(x) = 10x \ (mod\ 10)$ (Ten-fold map), which has periodic points of all orders, transitive and preserves length (measure preserving).

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- Note: $1 < \frac{1}{x_2}$. Let a_2 be the integer part of $\frac{1}{x_2}$. Thus $\frac{1}{x_2} = a_2 + x_3$, for some $0 \leq x_3 < 1$, and $x_2 = \frac{1}{a_2+x_3}$. Hence, $x = a_0 + \frac{1}{a_1+\frac{1}{a_2+x_3}}$.
- Similarly, $1 < \frac{1}{x_3}$; so, if a_3 be the integer part of $\frac{1}{x_2}$, we have $x = a_0 + \frac{1}{a_1+\frac{1}{a_2+\frac{1}{a_3+x_4}}}$. Continue this process ...

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Therefore, we obtain the “fractional” expression

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Infinite repeating CFE: $[a_0; a_1, a_2, \dots, a_k, \overline{a_{k+1}, \dots a_{k+l}}]$

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Question. Which type of real numbers have finite, infinite repeating, or infinite non-repeating CFEs?

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The integer r_k is the gcd of a and b.

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Divide both sides of the equations above by their quotients:

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- $r_{k-2} = a_k r_{k-1} \Rightarrow \frac{r_{k-2}}{r_{k-1}} = a_k$, which gives the CFE
  $$\frac{a}{b} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{r_3}{\ddots \cfrac{1}{a_{k-1} + \frac{1}{a_k}}}}}} = [a_0; a_1, a_2, \dots, a_k]!$$

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To answer this question we need some more work.

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Fact. Let $p_{-1} = 1$, $q_{-1} = 0$, $p_0 = a_0$ and $q_0 = 1$. Then $\forall\ k \geq 2$,

(*) $\quad p_k = a_k p_{k-1} + p_{k-2}$ and $q_k = a_k q_{k-1} + q_{k-2}$.

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Proof: Exercise. (Hint: Use induction.)

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Question. What can we say about the sequence $\{\frac{p_n}{q_n}\}_n$?

Convergents, cont.

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Note. $[a_0; a_1, a_2, \ldots] = \lim_n \frac{p_n}{q_n} = \lim_n [a_0; a_1, a_2, \ldots, a_n]$. Another manifestation of the fact $\overline{\mathbb{Q}} = \mathbb{R}$!

Periodic CFEs

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Is this a coincidence?

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- Since the CFE is infinite, θ cannot be rational; hence, it is a quadratic irrational.

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- Since the CFE is infinite, θ cannot be rational; hence, it is a quadratic irrational. Since $x = [a_0; a_1, \dots, a_{k-1}, \theta]$, it is also a quadratic irrational.

CFEs of quadratic irrationals

A *quadratic irrational* is a (non-rational) root of a quadratic polynomial equation with integer coefficients.

If $x = [a_0; a_1, \ldots, a_{k-1}, b_0, b_1, \ldots, b_{n-1}, b_0, b_1, \ldots, b_{n-1}, \ldots]$,

- Let $\theta = [b_0, b_1, \ldots, b_{n-1}, b_0, b_1, \ldots, b_{n-1}, \ldots]$. Then $\theta = [b_0; b_1, \ldots, b_{n-1}, \theta]$.
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Fact. $x \in \mathbb{R}$ has periodic CFE iff it is a quadratic irrational.

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Hence, continued fractions provide much efficient approximation of real numbers by rationals!

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- Thus $x = [a_0; a_1, a_2, \dots, a_k, \dots]$ is transcendental.

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THANK YOU!