DYNAMICS OF CONTINUED FRACTIONS

Doğan Çömez

NDSU
Math Club

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Outline:

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1. Dynamics of Ten-fold Map

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2. Continued Fraction Expansion

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3. The Gauss map

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3. The Gauss map
4. Dynamical Properties of the Gauss Map

Decimal expansion and Ten-fold map

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$x \in \mathbb{Q}^c$ iff x has non-repeating decimal expansion.

b) Some rationals have more than one different decimal expansions.

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Ten-fold map $T : [0,1) \to [0,1)$, where $T(x) = 10x (\text{mod } 1)$
$(Tx = 10x - i$ if $x \in [\frac{i}{10}, \frac{i+1}{10}),\ 0 \leq i \leq 9)$.

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- $a_k = [10T^{k-1}x],\ k = 0,1,2\dots$
- $T(0.a_1a_2a_3\dots a_n\dots) = 0.a_2a_3a_4\dots a_{n+1}\dots$
- No decimal expansion with infinite $999\dots$ occur. Hence, non-uniqueness problem is settled!

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- T has periodic points of all orders (points with $T^n x = x$). Indeed $x \in \mathbb{Q}$ iff it is a periodic point of T.
- $\exists \; x \in [0, 1)$ such that the set $\{T^k x\}_{k=0}^{\infty}$ is dense in $[0, 1)$. (i.e., T is *transitive*). Since irrational points are not periodic, the transitive points are among irrationals.

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  $\Sigma = \{(a_n); a_n \in \{0,1,2,\ldots,9\}\}$, and
  $\sigma(a_1, a_2, \ldots, a_n, \ldots) = (a_2, a_3, \ldots, a_{n+1}, \ldots)$ (*the shift map*).

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$$0.a_1a_2a_3\ldots a_n\cdots \in [0,1) \longleftrightarrow (a_1a_2a_3\ldots a_n\ldots) \in \Sigma.$$

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$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots a_{k-1} + \cfrac{1}{a_k + \dots}}}}} = [a_1, a_2, a_3, \dots, a_k, \dots]^*,$$

where $a_i \in \mathbb{Z}^+ \ \forall \ i \geq 0$ $(a_0 = 0)$.

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where $a_i \in \mathbb{Z}^+ \ \forall \ i \geq 0$ $(a_0 = 0)$.

Note: if $a_k = 0$ for some k, then $a_i = 0$ for all $i \geq k$; otherwise the process continues indefinitely!

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Examples. 1) $\frac{13}{20} = [1, 1, 1, 6]$
2) $\frac{\sqrt{10}-2}{3} = [\overline{21}]$
3) $\pi - 3 = [7, 15, 1, 292, 1, 1, 1, 2, 3, 1, 14, 4, \ldots]$

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- $\lim_{x \to \frac{1}{n}^+} G(x) = 1$ for $x \in (\frac{1}{n}, \frac{1}{n-1}]$, $n \geq 2$. Therefore, G is discontinuous at each rational point $\frac{1}{n}$, $n \geq 1$.

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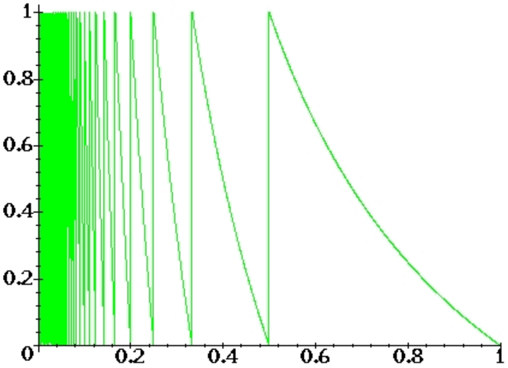
- G is monotone decreasing on each interval $(\frac{1}{n}, \frac{1}{n-1}]$, $n \geq 2$.
- For each $n \geq 2$, $G : (\frac{1}{n}, \frac{1}{n-1}] \to [0, 1)$ is 1-1 and onto.



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- if $a_2 = [\frac{1}{G(x)}]$; then $G^2(x) = \frac{1}{G(x)} - [\frac{1}{G(x)}] = \frac{1}{G(x)} - a_2$. Thus, $G(x) = \frac{1}{a_2 + G^2(x)}$; and hence, $x = \frac{1}{a_1 + \frac{1}{a_2 + G^2(x)}} = [a_1, a_2, G^2(x)]$.

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- Continuing in this manner, it follows that $x = [a_1, a_2, \ldots, a_{k-1}, a_k + G^k(x)]$.

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- Continuing in this manner, it follows that $x = [a_1, a_2, \ldots, a_{k-1}, a_k + G^k(x)]$.

Hence, for any $x \in [0, 1)$, we obtain its CFE by

$$(1) \quad x = [a_1, a_2, \ldots, a_k \ldots] := \lim_k [a_1, a_2, \ldots, a_{k-1}, a_k + G^k(x)].$$

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- If $x \in \mathbb{Q}$, then $G(x) \in \mathbb{Q}$. Also, $\exists k \geq 1$ s.t. $G^k(x) = 0$.
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Observe: *If* $\alpha = \frac{1}{a+\frac{1}{b}}$, then $\frac{1}{\alpha} = \frac{1}{(\frac{1}{a+\frac{1}{b}})} = \frac{1}{(\frac{b}{ab+1})} = \frac{ab+1}{b} = a + \frac{1}{b}$.

Hence $G([a,b]) = [b]$!

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Hence $G([a,b]) = [b]$! Therefore, if CFE of $x \in [0,1)$ is known,

$$(2) \quad G([a_1, a_2, \ldots, a_k, \ldots]) = [a_2, a_3, \ldots, a_k, \ldots].$$

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Fixed points of the system (i.e., $x \in [0, 1)$ with $G(x) = x$):

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Thus, G has infinitely many fixed points. Indeed, the point x with CFE $[\overline{k}]$ is the fixed point of G in the interval $[\frac{1}{k+1}, \frac{1}{k})$.

Fixed points

- $\frac{\sqrt{5}-1}{2} = [\overline{1}]$ is the fixed point in the interval $[1/2, 1)$.
- $\sqrt{2} - 1 = [\overline{2}]$ is the fixed point in the interval $[1/3, 1/2)$.
- $\frac{\sqrt{13}-3}{2} = [\overline{3}]$ is the fixed point in the interval $[1/4, 1/3)$.
- $\sqrt{5} - 2 = [\overline{4}]$ is the fixed point in the interval $[1/5, 1/4)$.
- $\frac{\sqrt{29}-5}{2} = [\overline{5}]$ is the fixed point in the interval $[1/6, 1/5)$.
- $\sqrt{10} - 3 = [\overline{6}]$ is the fixed point in the interval $[1/7, 1/6)$.
- $\frac{\sqrt{53}-7}{2} = [\overline{7}]$ is the fixed point in the interval $[1/8, 1/7)$.
- $\sqrt{17} - 4 = [\overline{8}]$ is the fixed point in the interval $[1/9, 1/8)$, etc.

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Question. Do you see any pattern here?

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- any point with CFE $[\overline{a_1, a_2, \ldots, a_k}]$ is periodic
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Observe: 1. Points with CFE of the form $[\overline{a_1, a_2, \ldots, a_k}]$ or $[a_1, \ldots, a_k, \overline{b_1, \ldots, b_k}]$ are dense in $[0,1)$. Hence,

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2. All periodic points are quadratic irrationals (no surprise)!

Periodic points, cont.

Some examples of periodic points:

- $\frac{\sqrt{10}-2}{3} = [\overline{2,1}]$ (periodic point),
- $\sqrt{11} - 3 = [\overline{3,6}]$ (periodic point),
- $\sqrt{3} - 1 = [\overline{1,2}]$ (periodic point),
- $\sqrt{6} - 2 = [\overline{2,4}]$ (periodic point),
- $\sqrt{7} - 2 = [\overline{1,1,1,4}]$ (periodic point),
- $\frac{\sqrt{30}-2}{13} = [3, \overline{1,2,1,4}]$ (eventually periodic point),
- $\frac{7-\sqrt{15}}{17} = [5, \overline{2,3}]$ (eventually periodic point).

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There is no example of $x \in [0,1)$ with $\overline{\{G^k(x)\}} = [0,1)$!

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Similarly, the system (Δ, σ), where $\Delta = \{(a_n); a_n \in \mathbb{Z}^+\}$ and σ is the shift, can be put into 1-1 correspondence with $([0,1), G)$ via the association

$$[a_1, a_2, a_3, \dots, a_n, \dots] \in [0,1) \longleftrightarrow (a_1a_2a_3\dots a_n\dots) \in \Delta.$$

However, this is not as useful as in the case of decimal system.

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But $m(0, \frac{1}{2}) = \frac{1}{2} \neq \ln(\frac{3}{2}) = m(G^{-1}(0, \frac{1}{2}))$! Bad news!

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Remark. The Gauss measure μ_G is “equivalent” to the (Lebesgue) measure m.

THANK YOU!