Existence and non-existence of ergodic Hilbert transform for admissible processes

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Department Colloquium, NDSU

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Solution: $u(x,y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(s)}{(x-s)^2+y^2} ds.$
Harmonic conjugate of $u(x,y)$ in the upper half plane is

$$v(x,y) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x-s) \frac{s}{s^2+y^2} ds.$$

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$$Hf(x) = \frac{1}{\pi} \text{ p.v.} \int_{-\infty}^{\infty} \frac{f(x-s)}{s} ds = \lim_{\epsilon \to 0} \int_{\epsilon < |s| < 1/\epsilon} \frac{f(x-s)}{s} ds.$$

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Besicovitch (early 1930's): *Hf* exists a.e. for all integrable f on $\mathbb{R}$.

Ergodic setting

Observe: Lebesgue measure is translation invariant; hence, the map $U_s(x) = x - s$ preserves Lebesgue measure, namely, $m(U_s^{-1}E) = m(E)$.

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More generally, if $\tau = \{T_t\}_{t \in \mathbb{R}}$ is a group of invertible m.p.t's on (X, Σ, μ), then the *ergodic Hilbert transform (eHt) of f* is defined as

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Discrete version: If $T : X \to X$ is an i.m.p.t. (i.e., $\tau = \{T^n\}_{n \in \mathbb{Z}}$), the *(discrete) ergodic Hilbert transform* of f is

$$H^T f(x) = \lim_{n \to \infty} \sum_{0 < |k| \leq n} \frac{f(T^k x)}{k}.$$

Ergodic theorems

Let (X, Σ, μ) be a probability space. A m.pt. $T : X \to X$ is called *ergodic* if $T^{-1}E = E$ implies $\mu(E) = 0$ or $\mu(E) = 1$.

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Birkhoff's Ergodic Theorem. (1931)

If $T : X \to X$ is m.p.t., then, $\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k x) = f^*(x)$ exists a.e. for all $f \in L_1$. If, furthermore, T is ergodic, then $f^* = \int f$.

Continuous parameter version:

Local Ergodic Theorem. (Wiener, 1939)

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Question: Analogous results for the (discrete or local) eHt?

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Question. Does eHt exist along moving averages sequences? We will return!

Admissible processes

$F = \{f_t\}_{t\in\mathbb{R}} \subset L_p$ is called a *τ-admissible process on* $\mathbb{R}$ if

$$T_s f_t \leq f_{s+t} \quad \text{and} \quad T_{-s} f_{-t} \leq f_{-s-t} \quad \text{for } t, s \geq 0.$$

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An admissible process $F = \{f_t\}_{t\in\mathbb{R}}$ is called

positive if $f_t \ge 0$ for all $t \in \mathbb{R}$,

strongly bounded if $\gamma_F := \sup_{t\in\mathbb{R}} \|f_t\|_p < \infty$,

symmetric if $T_{2t} f_{-t} = f_t$ for all $t \in \mathbb{R}^+$.

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Question. How about extending the **local** eHt result to the admissible processes?

LeHt for admissible processes-set up

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- M.A. Akcoglu & U. Krengel (1981) - $\exists$ a maximal τ-additive process $\{g \circ T_t\}$ such that $g \circ T_t(x) \leq f_t(x)$ a.e. for all $t \in \mathbb{R}$. (Hence, we can assume a given admissible F a positive process.)
- $\exists$ a family of non-negative integrable functions $\{u_t\}$ s.t. $f_t = T_t u_{|t|}$ for all $t \in \mathbb{R}$

LeHt for admissible processes-set up

Recall: Local eHt of f is

$$Hf(x) = \lim_{\epsilon\to 0} \int_{\epsilon\leq|t|\leq 1/\epsilon} \frac{f(T_s x)}{s} ds.$$

Local ergodic Hilbert transform of a symmetric admissible process $F = \{f_t\}_{t\in\mathbb{R}}$:

$$HF(x) = \lim_{\epsilon\to 0} \int_{\epsilon\leq|t|\leq 1/\epsilon} \frac{f_s(x)}{s} ds.$$

Let $F = \{f_t\}$ be a positive, symmetric, strongly bounded τ-admissible process. Then:

- M.A. Akcoglu & U. Krengel (1981) - $\exists$ a maximal τ-additive process $\{g \circ T_t\}$ such that $g \circ T_t(x) \leq f_t(x)$ a.e. for all $t \in \mathbb{R}$. (Hence, we can assume a given admissible F a positive process.)
- $\exists$ a family of non-negative integrable functions $\{u_t\}$ s.t. $f_t = T_t u_{|t|}$ for all $t \in \mathbb{R}$
- $\exists\ \delta \in L_1$ s.t. $f_t \leq T_t\delta, \forall\ t \in \mathbb{R}$, and $\|\delta\|_1 = \sup_{t\in\mathbb{R}} \|f_t\|_1$ (exact dominant of F).

LeHt for admissible processes-results

Let $\tau = \{T_t\}_{t\in\mathbb{R}}$ be an i.m.p. flow and $F = \{f_t\} \subset L_1$ be a symmetric, strongly bounded τ-admissible process with exact dominant δ.

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Theorem.1 (Maximal Inequality)

For any $\lambda > 0$, there exists a constant C such that,

$$\mu\{x : \sup_{q>0} |\int_{q\leq|s|\leq 1/q} \frac{f_s(x)}{s} ds| > \lambda\} \leq \frac{C}{\lambda}\|\delta\|_1.$$

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Theorem.2 (Local eHT for admissible processes)

$$HF(x) = \lim_{q\to 0} \int_{q\leq|s|\leq 1/q} \frac{f_s(x)}{s} ds \text{ exists a.e.}$$

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Ergodic version:

$$(^*)\quad H^u f(x) := \lim_{q\to 0}\int_{q<|s|\le a}\frac{f(T_s x)}{s}ds \;\text{ exists a.e. for all } f\in L_1,\; a>0.$$

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Hence: existence of the local eHt and (*) implies that

$$(**)\quad H^l f(x) := \lim_{q\to 0}\int_{a\le|s|<1/q}\frac{f(T_s x)}{s}ds \ \text{ exists a.e. for all } f\in L_1,\ a>0.$$

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  Then: G is τ-admissible, $G_t \leq F_t$ for all $t \in \mathbb{R}$, and $\gamma_G > \gamma_F - \epsilon$.

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Also, $H_q F(x) = H_q Z(x) + \int_{q \leq |s| \leq t_0} \frac{u_0(T_s x)}{s} ds + \int_{t_0 \leq |s| \leq 1/q} \frac{u_{t_0}(T_s x)}{s} ds$.

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Since the last two limits exist a.e. by (*) and (**), it remains to prove that $\lim_{q \to 0} H_q Z(x)$ exists a.e.

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Let $E = \{x : \limsup_q H_q F(x) - \liminf_q H_q F(x) > \lambda\}$, then

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Thus $HF(x) = \lim_{q \to 0} H_q F(x)$ exists a.e.

Moving averages

Moving averages sequences.

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A sequence $\mathbf{w} = \{(v_n, r_n)\}$ in $\mathbb{Z} \times \mathbb{Z}^+$, where $r_n > 0$, satisfies *the cone condition (CC)* if $\exists$ C such that $\forall$ $s \in \mathbb{R}^+$, $|\Omega_\alpha(s)| \leq Cs$, where

$$\Omega_\alpha = \{(t,s) : \exists n,\ |t - v_n| \leq \alpha(s - r_n)\}, \quad \text{and} \quad \Omega_\alpha(s) = \{x \in \mathbb{R} : (x,s) \in \Omega_\alpha\}.$$

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Question. How about the analogous results for the eHt?

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Let $T : X \to X$ be an i.m.p.t. and $\mathbf{w} = \{(v_n, r_n)\}$. Define the moving eHt of f by
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- Convergence on a dense subset of L_p?

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Let $f : X \to \mathbb{R}$ be such that $f(0) = f(2) = 0$ and $f(1) = 1$.

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Let $(X, \mathcal{F}, \mu, T)$, where $X = \{0,1,2\}$, $\mathcal{F} = \mathcal{P}(X)$, $\mu(\{i\}) = 1/3$, $0 \le i \le 2$, and $T(0) = 1$, $T(1) = 2$, $T(2) = 0$.

Let $\mathbf{w} = \{(n,n)\}$. $\mathbf{w}$ satisfies the cone condition and $v_n = n = r_n \to \infty$.

Consider two subsequences: $\mathbf{w}_1$ consists of those that are multiples of 3, $\mathbf{w}_2$ consists of those that are of the form $n = 3k+1$.

Let $f : X \to \mathbb{R}$ be such that $f(0) = f(2) = 0$ and $f(1) = 1$.

$$Hf(1) = \sum_{k \neq 0} \frac{f(T^k 1)}{k} = 0,$$

$$Hf(0) = \sum_{k \neq 0} \frac{f(T^k 0)}{k} = (1 - \frac{1}{2}) + (\frac{1}{4} - \frac{1}{5}) + (\frac{1}{7} - \frac{1}{8}) + \cdots \neq 0.$$

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Question. Is it possible to have an affirmative answer under some additional conditions?

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Theorem (Çömez, 2016) Let $\{(v_n, r_n)\}$ satisfy CC. If T has Lebesgue spectrum, then, for all $f \in L_2^0$, $\lim_n H_{(v_n, r_n)} f(x)$ exists a.e.

THANK YOU!