Modern ergodic theory; from a physics hypothesis to a mathematical theory

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If the energy of the system is E, then $\mathbf{x}$ must lie on the energy surface $H(\mathbf{x}) = E$, where H is the Hamiltonian

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}, \quad 1 \leq i \leq N.$$

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By Liouville's Theorem, τ preserves the normalized Lebesgue measure on X.

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- $(X, \mathcal{B}, \mu, \tau)$ is a measurable as well as topological dynamical system.

If $f : X \to \mathbb{R}$ denotes a function of a physical quantity, measured during an experiment, for any $t \geq 0$, $f(T_t\mathbf{x})$ is the value it takes at the instant t, provided that the system is at $\mathbf{x}$ when $t = 0$.

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Since macroscopic interval of time for the measurements is extremely large from the microscopic point of view, one may actually consider the limit of the time averages:

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t f(T_t\mathbf{x})dt.$$

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This is the *ergodic hypothesis* of Boltzmann.

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An interesting talk by CalTech physicist Sean Carroll in YouTube.

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The orbit of a point is dense in the phase space. This is a reasonable assumption, which is accepted as the actual and workable hypothesis by adherents of the theory and many mathematicians.

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1. Which dynamical systems satisfy the ergodic hypothesis?
2. In a dynamical system, can we always expect the time average be equal to space average?
3. What is the structure of dynamical systems satisfying the ergodicity?

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Will consider a discrete group of transformations: $\tau = \{T^n\}_{n \in \mathbb{Z}}$, where $T : X \to X$ is be a measure preserving transformation (i.e., $\mu(E) = \mu(T^{-1}E)$ for all $E \in \mathcal{B}$).

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Consequently, the ergodic hypothesis of Boltzmann becomes

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} f(T^k x) = \int_X f(x)d\mu(x).$$

Recurrence

Theorem. (Poincare Recurrence Theorem, 1893) If $E \in \mathcal{B}$ with $\mu(E) > 0$, then for almost every $x \in E$ there exists $k \geq 1$ such that $T^k x \in E$.

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So, Q, $T^{-1}Q$, $T^{-2}Q$, $\ldots$, $T^{-n}Q, \ldots$ are all pairwise disjoint, each having measure $\mu(Q)$. Since $\mu(X) = 1$, this is possible only if $\mu(Q) = 0$, equivalently, $\mu(E) = \mu$(recurrent points). □

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Definition. A mpt $T : X \to X$ is *ergodic* if $E \in \mathcal{B}$ with $T^{-1}E = E$, then $\mu(E) = 0$ or 1.

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Example 1. (Irrational rotation) Let $X = S$ be the unit circle as above. Let $T : S \to S$ be defined by $Tz = e^{\alpha 2\pi i} z$, where $\alpha \in (0, 1)$ is an irrational number. This is a very rigid system, in the sense that, the orbit of *every point* is dense in S.

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a) $\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k x) = f^*(x)$ exists for almost every $x \in X$,

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Note: the assertions may not hold on a set of measure zero. This may be a concern for philosophically minded, since there are sets that are topologically dense in X while having measure zero.

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Theorem. (Dunford-Schwartz, 1956) Let $(X, \mathcal{F}, \mu)$ be a probability space and $T : L_p(X) \to L_p(X)$ be a linear operator with $\|T\|_1 \leq 1$ and $\|T\|_\infty \leq 1$. Then, for all $f \in L_1(X)$,

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There are many other operator theoretical generalizations as well.

Multiparameter ergodic theorems

Assume $T : X \to X$ and $S : X \to X$ be two mpts, or $T, S : L_p(X) \to L_p(X)$ be linear contractions. Then, it make sense to consider averages of the form $\frac{1}{mn} \sum_{i,j=1}^{m,n} T^i S^j f(x)$.

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Theorem. (Brunel, 1973) If T and S commute, then for all $f \in L_1(X)$, $\lim_n \frac{1}{n^2}\sum_{i,j=1}^{n} T^i S^j f(x)$, exists a.e..

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Extension to commuting positive linear contractions is due to Ornstein (1973) and Akcoglu-del Junco (1975).

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The answer is affirmative.

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- The sequence $[n^{3/2}]$ (in $L_p,\ 1 < p < \infty$)
- The sequence $[n \log n]$ (in $L_p,\ 1 < p < \infty$)
- Return time sequences (in L_1)
- Randomly generated sequences of positive density (in L_1)
- Randomly generated sequences of zero density (in L_2)
- Sequences of squares (in $L_p,\ 1 < p < \infty$)
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Furthermore, the limit along the first three sequences is the right one, namely, it is the space average!

Convergence along moving averages

What if the measurements are made along a sequence like

$$v_1, v_1+1, v_1+2, \dots, v_1+r_1,$$
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$$\dots$$
$$v_n, v_n+1, v_n+2, \dots, v_n+r_n, \quad \text{and so on,}$$

where $v_n \uparrow,\ r_n \uparrow$ and $v_n + r_n < v_{n+1}$.

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If the sequence $\{(v_n, r_n)\}_n$ satisfies a condition called cone condition, then

$$\lim_n \frac{1}{r_n} \sum_{k=0}^{r_n} f(T^{v_n+k}x) \text{ exists a.e. for all } f \in L_1.$$

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- $\{a_k\}$ is a bounded Besicovitch sequence

Modulated ergodic theorems

What if the measurements are somewhat "tainted", or modulated? That is, instead of obtaining the values $f(T^k x)$ along the orbit of the point x, we would be getting values like $a_k f(T^k x)$ for some sequence $\{a_k\}$. Then, we'll be end up with *modulated* averages: $\frac{1}{n}\sum_{k=0}^{n-1} a_k f(T^k x)$.

Question. For which sequences $\{a_k\}$ does $\lim_n \frac{1}{n}\sum_{k=0}^{n-1} a_k f(T^k x)$ converge a.e.?

A list of modulating sequences $\{a_k\}$ along which a.e. convergence holds, i.e., $\lim_n \frac{1}{n}\sum_{k=0}^{n-1} a_k f(T^k x)$ exists a.e. is:

- $a_k = \lambda^k$, where $|\lambda| = 1$
- $\{a_k\}$ is a bounded Besicovitch sequence
- $\{a_k\}$ is a sequence having a mean.

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Notice that T does not map $[0,1]$ into itself. If $x \in (\frac{1}{3}, \frac{2}{3})$, then $Tx \notin [0,1]$. So, $J_1 = \{x \in [0,1] : Tx > 1\} = (\frac{1}{3}, \frac{2}{3})$.

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Define, $J_2 = \{x \in [0,1] : T^2x > 1\}$. Thus, $T^2([0,1]) = [0,1] \setminus (J_1 \cup J_2)$ consists of four closed intervals that are mapped one-to-one and onto $[0,1]$.

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Taking the set of all such intervals, define $\mathcal{C} = \{x \in [0,1] : T^n x \in [0,1],\ \forall n \geq 1\} = \cap_{n=1}^{\infty} T^n([0,1])$. Since all of these intervals are one-to-one and onto $[0,1]$, $\mathcal{C}$ is mapped to itself. Hence, the pair $(\mathcal{C}, T)$ is a dynamical system.

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Furthermore, from the construction, we can see that $\mathcal{C} = \bigcap_{n \geq 1} I_n$. This is the **Cantor set**, a *fractal!*

One can, of course, construct the Cantor set via a “static” manner. However, this dynamic construction paves way to introduce many tools of dynamical systems into study of properties of fractals.

One can, of course, construct the Cantor set via a “static” manner. However, this dynamic construction paves way to introduce many tools of dynamical systems into study of properties of fractals. Indeed, some fundamental features of fractals (such as box dimension, Hausdorff dimension, subfractal structure, etc.) have been studied in depth only after the introduction of ergodic theory tools into fractal geometry.

THANK YOU!

Some sources used

Some sources used

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