ORDER IN CHAOTIC DYNAMICAL SYSTEMS

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- solar system
- the stock market
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Some of these systems are highly regular, such as the solar system, or the simple pendulum, and some are irregular.

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Another (very interesting) example: The Logistic system (representing population behavior).

Logistic system

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Logistic Equation

If T represents time from one instance to the next, this equation can be rewritten as $T(x) = rx(1-x)$, where $x \in [0, 1]$.

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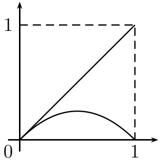

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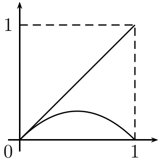

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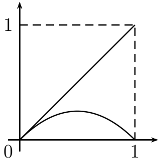

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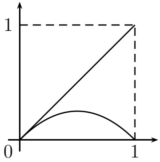

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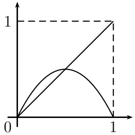

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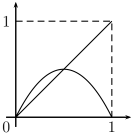

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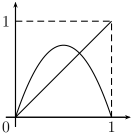

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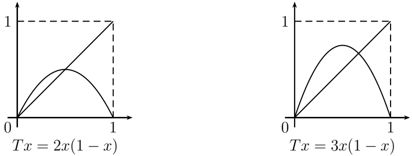

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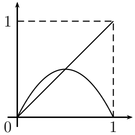

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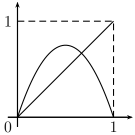

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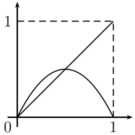

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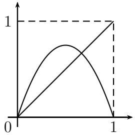

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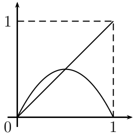

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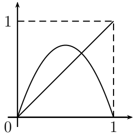

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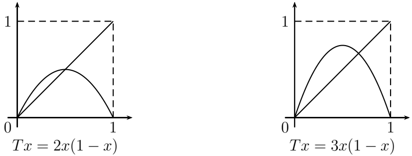

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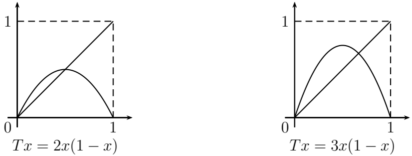

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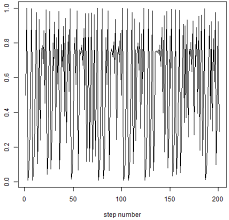

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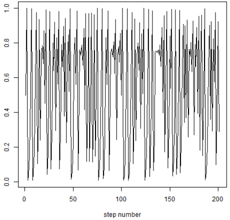

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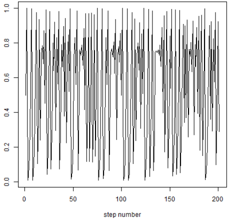

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LET'S HAVE SOME FUN!

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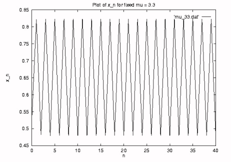

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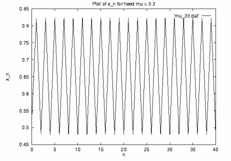

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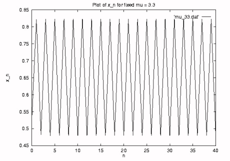

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$1 + \sqrt{6} < r \leq 3.54$ (approx.) : the orbits of almost all points oscillate among four values,

$r > 3.54$ (approx.) : the orbits of almost all points oscillate among 8 values, then 16, 32, and so on.

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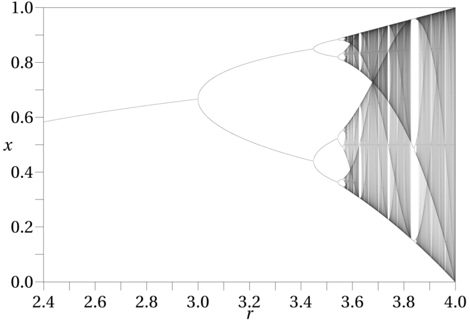

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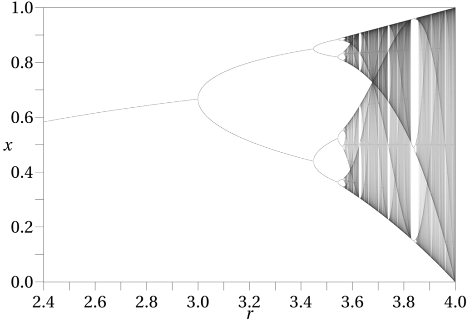

CHAOS!**

Tent map

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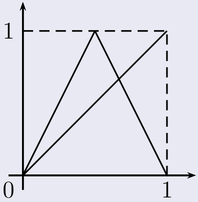
Tent map

For $r = 4$, the Logistic system is equivalent to the **Tent system:**



$$Tx = \begin{cases} 2x, & \text{if } 0 \le x < \frac{1}{2} \\ 2(1-x), & \text{if } \frac{1}{2} \le x \le 1 \end{cases}$$

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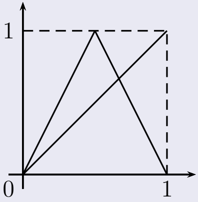

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This equivalence is given by the (invertible) map $\phi(x) = \sin^2(\frac{\pi x}{2})$.

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- 1

Tent map

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Do you see a pattern here?

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Do you see a pattern here? Are there other periodic points? **

Tent map

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Fact.1

If $0 < x \leq 1$ is rational, then it is eventually periodic.

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Theorem (Li and Yorke)

If $f : [0,1] \to [0,1]$ is a continuous map and if it has a periodic point of period 3, then it has periodic points of any period k.

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If $f : [0,1] \to [0,1]$ is a continuous map and if it has a periodic point of period 3, then it has periodic points of any period k.

Hence, the Tent map (equivalently, the Logistic map) has periodic points of any period $k \geq 1$!

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Fact.2

If $0 < x < 1$ is irrational, then the orbit $\{T^n x\}$ has an element arbitrarily close to any point $y \in [0, 1]$.

That is, the Tent (Logistic) map is **topologically transitive** (i.e., has a point whose orbit is dense).

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- is topologically transitive. **

Symbolic dynamics

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Symbolic dynamics

Let $A = [0, \frac{1}{2}]$, and $B = (\frac{1}{2}, 1]$. Give an *address* to each $x \in [0,1]$:

$$T^n x \leftrightarrow 0 \text{ if } T^n x \in A \text{ and } T^n x \leftrightarrow 1 \text{ if} T^n x \in B.$$

Hence, we have *

- $0 = (0, 0, 0, \dots)$ since $T^n(0) \in A$ for $n \geq 0$.
- $\frac{2}{3} = (1, 1, 1, \dots)$ since $T^n(\frac{2}{3}) \in B$ for $n \geq 0$.
- $1 = (1, 0, 0, 0, \dots)$ since $T(1) \in A$ for $n \geq 1$.
- $\frac{1}{2} = (0, 1, 0, 0, \dots)$ since $T(\frac{1}{2}) \in B$; $\frac{1}{2}, T^n(\frac{1}{2}) \in A$ for $n \geq 2$.
- $\frac{1}{4} = (0, 1, 1, 0, 0, 0, \dots)$ since $T^n(\frac{1}{4}) \in B$ if $n = 1, 2$ and $T^n(\frac{1}{4}) \in A$ for $n \geq 3$.
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- A transitive point:

$$(0\ 1\ 00\ 01\ 10\ 11\ 000\ 001\ 010\ 100\ 011\ 101\ 110\ 111\ 0000\ 0001\ \ldots) \quad **$$

Back to order

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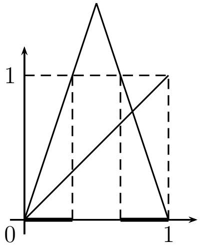
- Attractors can be a single point ($r = 1$ in Logistic map), a finite set ($1 < r < 3.5$ in Logistic map) or more complicated sets (Tent map). When the attractor is not expressible as countable union or intersection of geometric objects (like points, lines, surfaces, etc.), it is called a **strange attractor**.

Cantor set

Tent map $T : [0,1] \to [0,1]$, given by the equation



$$Tx = \begin{cases} 3x, & \text{if } 0 \le x < \frac{1}{2} \\ 3(1-x), & \text{if } \frac{1}{2} \le x \le 1 \end{cases}$$

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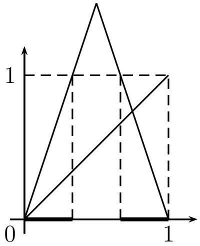

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Actual domain of T is $T^{-1}[0, 1] = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$.

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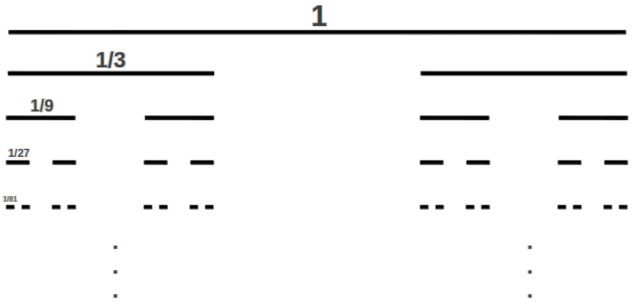

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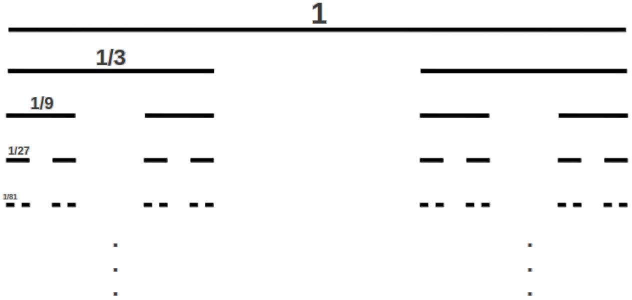

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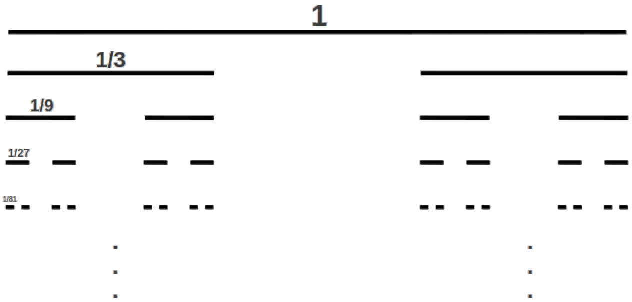

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- Many fractals are strange attractors of some dynamical systems.

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THANK YOU!