FUNDAMENTALS OF REAL ANALYSIS
by

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Background: All of Math 450/1 material. Namely: basic set theory, relations and PMI, structure of $\mathbb{N}, \mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$, basic properties of (continuous and differentiable) functions on $\mathbb{R}$, cardinality, Riemann integral, sequences of real numbers, sequences of functions, pointwise/uniform convergence.

OUTLINE OF THE COURSE

I. Review of $\mathbb{R}$ and metric spaces
II. Lebesgue measure and measure spaces
III. Measurable functions and Lebesgue integration
IV. Differentiation and signed measures
V. Product measure spaces
VI. Special topics (Lp-spaces, modes of convergence and Hausdorff dimension)

REFERENCES

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Real and Abstract Analysis, by E. Hewitt & K. Stromberg; **Springer-Verlag**, 1975.

Introductory Real Analysis, by A. Kolmogorov & S.V. Fomin; **Dover**, 1975.

Real Analysis, by H.L. Royden & P.M. Fitzpatrick; 4th ed., **Prentice Hall**, 2010.

I. REVIEW OF THE REAL NUMBER SYSTEM AND METRIC SPACES

I.1. Axiomatic construction of $\mathbb{R}$.

The real number system is a complete ordered field, i.e., it is a set $\mathbb{R}$ which is endowed with addition and multiplication operations satisfying the following axioms

1. $(\mathbb{R}, +, .)$ is a field with additive identity 0 and multiplicative identity 1.
2. $(\mathbb{R}, \leq)$ is a partially ordered set compatible with the field axioms in the sense that

(i) $\forall x, y \in \mathbb{R}$, either $x \leq y$ or $y \leq x$,

(ii) if $x \geq y$, then $x + z \geq y + z$ for all $z \in \mathbb{R}$,

(iii) if $x \geq y$ and $z \geq 0$, then $xz \geq yz$.

3. (Completeness axiom) Every nonempty set of real numbers bounded above has a least upper bound (supremum). [Alternative statement: Every nonempty set of real numbers bounded below has a greatest lower bound (infimum).]

Remarks. 1. Any complete ordered field is isomorphic to $(\mathbb{R}, +, ., \leq)$ with Completeness axiom.

2. For a set $S \subset \mathbb{R}$, $\sup(S)$ need not belong to S.
3. If $\sup(S)$ exists, it is unique. (Hence we have set functions!)
4. Since $\inf(-S) = -\sup(S)$, corresponding statements also hold for $\inf$.
5. $\mathbb{R}$ is unbounded (hence so are $\mathbb{N}, \mathbb{Z}$ and $\mathbb{Q}$).
6. For any $x, y \in \mathbb{R}$, define *maximum* and *minimum of x and y* by

$$x \vee y = \begin{cases} x \text{ if } x \geq y \\ y \text{ if otherwise} \end{cases} \quad \text{and} \quad x \wedge y = \begin{cases} y \text{ if } x \geq y \\ x \text{ if otherwise,} \end{cases} \quad \text{respectively.}$$

Fact.1 If $x + \epsilon \geq y$ for all $\epsilon > 0$, then $x \geq y$.

Proof. Exercise.

Exercises. 1. If $a < c$ for all c with $c > b$, then $a \leq b$.

2. For $A \subset \mathbb{R}$ nonempty and $\alpha \in \mathbb{R}$ an upper bound for A, $\alpha = \sup(A)$ if and only if $\forall \epsilon > 0\ \exists x \in A$ such that $\alpha - \epsilon < x \leq \alpha$.

3. If $a \leq b\ \forall a \in A$, then $\sup(A) \leq b$.

4. If $A \subset \mathbb{R}$ is bounded (i.e. bounded above and below) and $B \subset A$ is nonempty, then

$$\inf(A) \leq \inf(B) \leq \sup(B) \leq \sup(A).$$

5. Let A and B be nonempty subsets of $\mathbb{R}$ such that $a \leq b\ \forall a \in A\ \forall b \in B$. Then (i) $\sup(A) \leq \sup(B)$ if $\sup(B)$ exists, and (ii) $\sup(A) \leq \inf(B)$.

6. Let A be a nonempty subset of $\mathbb{R}$ with $\alpha = \sup(A)$. If for $c \geq 0$, $cA := \{ca : a \in A\}$, then $c\alpha = \sup(cA)$.

7. Let A and B be nonempty subsets of $\mathbb{R}$ and let $C = \{a + b : a \in A,\ b \in B\}$. If $\sup(A)$ and $\sup(B)$ exist, then so does $\sup(C)$ and $\sup(C) = \sup(A) + \sup(B)$.

8. Exercises 2, 4, 6-8 are also is true for infima with obvious (appropriate) changes in the statements.

Fact.2 $\forall x \in \mathbb{R}\ \exists n \in \mathbb{Z}^+$ such that $n > x$.

Proof. Exercise.

Fact.3 (Archimedean Property) If $x, y \in \mathbb{R}$ with $x > 0$, then $\exists n \in \mathbb{Z}$ such that $y < nx$.

Proof. Exercise.

Corollary. a) If $x, y \in \mathbb{R}$ with $x < y$, then $\exists z \in \mathbb{Q}$ ($\mathbb{Q}^c$) such that $x < z < y$.

b) $\forall x > 0\ \exists n \in \mathbb{Z}^+$ such that $n - 1 \leq x < n$.

c) $\forall \epsilon > 0\ \exists n \in \mathbb{Z}^+$ such that $\frac{1}{n} < \epsilon$.

Proof. Exercise.

Below are some important inequalities on (finite) sequences of real numbers: if $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ are real numbers, then

$$(\sum_{k=1}^{n} a_k b_k)^2 \leq (\sum_{k=1}^{n} a_k^2)\ (\sum_{k=1}^{n} b_k^2) \quad \text{(Cauchy-Schwartz Inequality), and}$$

$$(\sum_{k=1}^{n} |a_k + b_k|^2)^{1/2} \leq (\sum_{k=1}^{n} a_k^2)^{1/2} + (\sum_{k=1}^{n} b_k^2)^{1/2} \quad \text{(Minkowski's Inequality).}$$

I.2. The Extended Real Number System.

The set $\mathbb{R}^{\#} := \mathbb{R} \cup \{\mp\infty\}$ with operations

(i) $x + \mp\infty = \mp\infty$ for all $x \in \mathbb{R}$; $x(\mp\infty) = \mp\infty$ if $x > 0$ and $(-x)(\mp\infty) = \pm\infty$

(ii) $\infty + \infty = \infty$, $\ (-\infty) + (-\infty) = -\infty$; $\ \infty.(\mp\infty) = \mp\infty$; $\ 0.(\mp\infty) = 0$

and with the order property that $-\infty < x < \infty$ for all $x \in \mathbb{R}$, is called the **extended real number system.**

Remarks. 1) $\infty - \infty$ is not defined.

2) $\sup(\emptyset) = -\infty$ is assumed.

3) If a set A of real numbers has no upper bound, then we say $\sup(A) = \infty$. Hence, in $\mathbb{R}^{\#}$ every set has a supremum (infimum).

I.3. The topology of $\mathbb{R}$.

In the next three sections, we will provide a short review of some important concepts and facts from real analysis. For the proofs of most of the statements one can refer to any one of the references listed above. Of course, its is strongly advised that the reader should try to provide proofs on her/his own.

For any $x, y \in \mathbb{R}$, define the *absolute value of* x by $|x| = \begin{cases} x \text{ if } x \geq 0 \\ -x \text{ if } x < 0, \end{cases}$ which can be interpreted as *the distance* of x to 0. Now, for any $a \in \mathbb{R}$ **the (open) ball centered at** a **with radius** $r > 0$ is the set

$$B(a, r) := \{x \in \mathbb{R} : |x - a| < r\} = (a - r, a + r).$$

For a set $A \subset \mathbb{R}$, a point $a \in A$ is said to be an *interior* point if there exists $\epsilon > 0$ such that $B(a, \epsilon) \subset A$. The set of all interior points of a set A is called the *interior of* A and is denoted by $int(A)$. A set $A \subset \mathbb{R}$ is called *open* if every point of it is an interior point. A set $A \subset \mathbb{R}$ is called *closed* if A^c is open. The empty set is assumed open (and closed).

Fact. 4 a) The union (intersection) of any collection of open (closed) sets is open (closed).

b) The union (intersection) of any finite collection of closed (open) sets is closed (open).

Fact. 5 Every nonempty open set of real numbers is a disjoint countable union of open intervals.

A real number p is called an *accumulation point* of a set $A \subset \mathbb{R}$ if for every $\epsilon > 0$ we have $B(p, \epsilon) \cap A \neq \emptyset$. The set of all accumulation points of a set A is called the *derived set of* A and is denoted by A'. The smallest closed set containing a set $A \subset \mathbb{R}$ is called the *closure of* A and is denoted by $\bar{A}$.

Theorem. (Bolzano-Weierstrass) Every bounded infinite subset of $\mathbb{R}$ has an accumulation point.

Fact. 6 a) For any $A \subset \mathbb{R}$, $int(A) \subset A \subset \bar{A}$.

b) A set $A \subset \mathbb{R}$ is closed iff $A = \bar{A}$.
c) For any $A \subset \mathbb{R}$, $int(A)$ is the largest open set contained in A.
d) $\bar{\mathbb{Q}} = \mathbb{R}$. (Hence, $\mathbb{R}$ is *separable*, i.e. there exists a countable set $A \subset \mathbb{R}$ such that $\bar{A} = \mathbb{R}$.)

Exercises. Let $A, B \subset \mathbb{R}$, then:

1) $\bar{A} = A \cup A'$.

2) $int(A) \cup int(B) \subset int(A \cup B)$ and $int(A) \cap int(B) = int(A \cap B)$.

3) $\bar{A} \cup \bar{B} = \overline{A \cup B}$ and $\overline{A \cap B} \subset \bar{A} \cup \bar{B}$.

A collection $\mathcal{U} = \{U_\lambda : \lambda \in \Lambda\}$ of subsets of $\mathbb{R}$ is called a *cover for* $A \subset \mathbb{R}$ if $A \subset \cup_{\lambda \in \Lambda} U_\lambda$. The collection $\mathcal{U}$ is called an *open cover for* A if each U_λ is open.

Theorem. (Heine-Borel) Let $A \subset \mathbb{R}$ be a closed and bounded set and $\mathcal{U}$ be an open cover. Then there is a finite subcollection of $\mathcal{U}$ that covers A.

A set $C \subset \mathbb{R}$ is called a *compact set* if every open cover of it has a subcover consisting of finitely many elements.

Corollary TFAE:

a) $A \subset \mathbb{R}$ is compact.
b) A is closed and bounded.
c) Every infinite subset of A has an accumulation point in A.

Theorem. (Nested Set Property or Cantor Intersection Theorem) If $\{F_n\}$ is a collection of closed and bounded set of real numbers such that $F_n \subset F_{n-1}$ for all $n \geq 1$, then $\cap_{n \geq 1} F_n \neq \emptyset$.

I.4. Sequences of real numbers

Recall that a *sequence* is a function $a : \mathbb{N} \to \mathbb{R}$; for convenience, we denote sequences by $(a_n)_{n \geq 1}$, where $a(n) = a_n$, $n \geq 1$. By definition, a sequence is an infinite set of real numbers; hence, if bounded, it has at least one accumulation point, say a, by Bolzano-Weierstrass Theorem. If a sequence has only one accumulation point, it is called the *limit* of the sequence and the sequence is called *convergent* or we say that the sequence *converges to* a, and is denoted by $a_n \to a$ or $\lim_n a_n = a$. More explicitly,

$$a_n \to a \text{ if and only if } \forall \epsilon > 0 \; \exists N \in Z^+ \text{ such that } n \geq N \;\Rightarrow |a_n - a| < \epsilon.$$

Fact.7 Let $A \subset \mathbb{R}$. Then $a \in A'$ if there exists $(a_n) \subset A$ such that $a_n \to a$.

A sequence (a_n) is called a *Cauchy sequence* if $\forall \epsilon > 0 \; \exists N \in Z^+$ such that $m, n \geq N \;\Rightarrow |a_n - a_m| < \epsilon$.

Exercise. If (a_n) is a convergent (Cauchy) sequence, then it is bounded (i.e., $\exists M > 0$ such that $|a_n| \leq M$ for all n).

Fact.8 a) Every monotone bounded sequence of real numbers is convergent.

b) A sequence of real numbers is convergent if an only if it is Cauchy.
c) (Bolzano-Weierstrass Theorem for sequences, version 1) Every bounded sequence of real numbers has a convergent subsequence.

Remarks. 1) $\lim_n a_n = \infty$ means that $\forall c > 0\ \exists N > 0$ such that $n \geq N \Rightarrow a_n > c$. Similarly, $\lim_n a_n = -\infty$ means that $\forall c < 0\ \exists N > 0$ such that $n \geq N \Rightarrow a_n < c$.

2) In general, if we say $a_n \to a$, then $-\infty < a < \infty$; if $(a_n) \subset \mathbb{R}^\#$, then $-\infty \leq a \leq \infty$.

Observe that $a_n \to a$ means, for any $\epsilon > 0$, *all but finitely many* a_n's are in the interval $(a - \epsilon, a + \epsilon)$. A weakening of this is requiring *infinitely many* of a_n's are in $(a - \epsilon, a + \epsilon)$.

Definition. A real number a is called a *cluster point* (or *accumulation point*) of a sequence (a_n) if $\forall \epsilon > 0\ \exists$ infinitely many $a_n \in (a - \epsilon, a + \epsilon)$.

Equivalently, a is a cluster point of (a_n) iff $\forall \epsilon > 0$ and $\forall m \in \mathbb{Z}^+$, $\exists n \geq m$ such that $a_n \in (a - \epsilon, a + \epsilon)$.

Examples. 1) $a_n :\ 1,\ 1,\ 1/2,\ 1,\ 1/3,\ 1,\ 1/4,\ \ldots$. Accumulation points are 0, 1.

2) $((-1)^n)$. Accumulation points are -1, 1.

3) $(\frac{1}{\ln n})_{n=2}^{\infty}$. Accumulation point is 0.

Remark. A sequence (a_n) may have more than one accumulation points. In that case, it is not a convergent sequence; but, for each accumulation point, it has a subsequence convergent to that accumulation point (Exercise). In particular, if $a_n \to a$, then a is an accumulation point (the only one).

Theorem. (Bolzano-Weierstrass Thoerem for sequences, version 2) Every bounded sequence of real numbers has an accumulation point.

Question. How do we know that a given sequence has a limit?

Fact.9 Every bounded monotone sequence of real numbers is convergent.

Proof. (Sketch) By Bolzano-Weierstrass Thoerem for sequences, version 2, the sequence has an accumulation point. Since it's monotone, it has only one accumulation point; hence, it must be convergent. ∎

Fact.10 (Cauchy Criterion for sequences) A sequence of real numbers (a_n) is convergent if an only if it is a Cauchy sequence.

Question. Can we associate a real number to **any** (not necessarily convergent) sequence of real numbers?

First, recall that supremum and infimum of any bounded set of real numbers exit (if unbounded, they exist in $\mathbb{R}^\#$). Now, given any sequence of real numbers (a_n), define, for $k \geq 1$,

$$\underline{a_k} = \inf\{a_k,\ a_{k+1},\ a_{k+2},\ \dots\} \text{ and}$$
$$\overline{a_k} = \sup\{a_k,\ a_{k+1},\ a_{k+2},\ \dots\}.$$

Then, it follows that $\underline{a_k} \le \underline{a_{k+1}}$ and $\overline{a_k} \ge \overline{a_{k+1}}$ for all $k \ge 1$. Hence, $\{\underline{a_k}\}$ is monotone increasing and $\{\overline{a_k}\}$ is monotone decreasing sequence. Therefore, $\lim_k \underline{a_k}$ and $\lim_k \overline{a_k}$ exist (in $\mathbb{R}^\#$). (If (a_n) is bounded, then $\lim_k \underline{a_k}$ and $\lim_k \overline{a_k}$ exist in $\mathbb{R}$.) Also, observe that $\lim_k \underline{a_k} = \sup_{k\ge 1} \inf_{n\ge k}\{a_n\}$, and $\lim_k \overline{a_k} = \inf_{k\ge 1} \sup_{n\ge k}\{a_n\}$.

Definition. For any sequence of real numbers (a_n), define $\limsup_n a_n$ and $\liminf_n a_n$ as

$$\limsup_n a_n = \overline{\lim_n} a_n = \inf_{k\ge 1} \sup_{n\ge k}\{a_n\}, \text{ and}$$
$$\liminf_n a_n = \underline{\lim}_n a_n = \sup_{k\ge 1} \inf_{n\ge k}\{a_n\}.$$

Remarks. 1. For all $n \ge 1$, $\underline{a_k} \le a_n \le \overline{a_k}$ by construction.

2. For any $i, j \ge 1$, we have $\underline{a_i} \le \underline{a_{i+j}} \le \overline{a_{i+j}} \le \overline{a_i}$; hence, $\liminf_n a_n \le \limsup_n a_n$.

3. $\liminf_n a_n = \limsup_n a_n$ if and only if (a_n) is convergent; in that case,

$$\liminf_n a_n = \limsup_n a_n = \lim a_n.$$

4. $\liminf_n(-a_n) = -\limsup_n a_n$.

Exercise. Prove that $\liminf_n a_n$ ($\limsup_n a_n$) is the smallest (largest) of all the limit points of the *set* $\{a_n\}$.

I.5. Brief review of real-valued continuous functions on $\mathbb{R}$.

Let $A \subset \mathbb{R}$ and $a \in A$. Recall that a function $f : A \to \mathbb{R}$ is **continuous at** a iff $\forall \epsilon > 0, \exists \delta > 0$ such that if $x \in B(a, \delta)$ then $f(x) \in B(f(a), \epsilon)$.

Fact.11 A function $f : A \to \mathbb{R}$ is continuous at $a \in A$ if and only if for any sequence $(a_n) \subset A$ with $a_n \to a$, $f(a_n) \to f(a)$.

Fact.12 A function $f : A \to \mathbb{R}$ is continuous on A if and only if for any open set $O \subset \mathbb{R}$, $f^{-1}(O)$ is (relatively) open in A.

The following theorems indicate the reason why we value continuous functions.

Theorem. (Extreme value theorem) Every continuous function $f : A \to \mathbb{R}$, where $A \subset \mathbb{R}$ is compact, attains both of its extrema.

Theorem. (Intermediate Value Theorem) If $f : [a, b] \to \mathbb{R}$ is continuous, where $-\infty < a < b < \infty$, and $f(a) < \gamma < f(b)$, then $\exists c \in (a, b)$ such that $f(c) = \gamma$.

Recall that a function $f : A \to \mathbb{R}$ is **uniformly continuous on** A if and only if $\forall \epsilon > 0, \exists \delta > 0$ such that if $|x - y| < \delta$ then $|f(x) - f(y)| < \epsilon$ for all $x, y \in A$. Note that not every continuous function is uniformly continuous; however, under some conditions this is true.

Theorem. If $f : A \to \mathbb{R}$ is continuous (on A) and A is compact, then f is uniformly continuous (on A).

Let $A \subset \mathbb{R}$ be a compact set. Define $C(A) = \{f : A \to \mathbb{R} : f \text{ is continuous}\}$. Hence, every $f \in C(A)$ is uniformly continuous. With the usual addition and scalar multiplication of functions, $C(A)$ is a vector space over $\mathbb{R}$. Furthermore, if for any $f, g \in C(A)$,

$$d(f,g) = \sup_{x\in A} |f(x) - g(x)|,$$

then the function $d : C(A) \times C(A) \to \mathbb{R}$ defines a metric on $C(A)$ (Exercise: Prove this fact); making it a metric space. Note that $d(f, g)$ is well-defined and is finite for any $f, g \in C(A)$. Hence, it makes sense to talk about *convergence* in the metric space $C(A)$.

Let (f_n) be a sequence in $C(A)$ and $f \in C(A)$. Recall that (f_n) is said to *converge pointwise to* f iff $\forall x \in A,\ \forall \epsilon > 0, \exists N(\epsilon, x)$ such that if $n \geq N$, then $|f_n(x) - f(x)| < \epsilon$. The sequence (f_n) is said to *converge uniformly to* f iff $\forall \epsilon > 0, \exists N(\epsilon)$ such that if $n \geq N$, then $|f_n(x) - f(x)| < \epsilon,\ \forall x \in A$.

Clearly, every uniformly convergent sequence is pointwise convergent; but the converse need not hold. However, under some (rather strong) conditions the converse holds (such as under the conditions of Dini's Theorem).

Recall that, in the metric space $(C(A), d)$, a sequence (f_n) converges to $f \in C(A)$ iff $\forall \epsilon > 0, \exists N(\epsilon)$ such that if $n \geq N$, then $d(f_n, f) < \epsilon$. It turns out that there is a strong connection between convergence in the metric d and uniform convergence (on A).

Theorem. A sequence (f_n) in $C(A)$ converges to $f \in C(A)$ iff $f_n \to f$ uniformly on A.

The metric space $C(A)$ has some other very desirable properties;

Fact.13 If $f \in C(A)$, then $\exists M > 0$ such that $|f(x)| \leq M$ for all $x \in A$. [Equivalently, $\exists M > 0$ such that $f \in B(\underline{0}, M)$, where $\underline{0}$ is the zero function.]

Theorem. The metric space $(C(A), d)$ is complete. [Hence, every Cauchy sequence of functions $(f_n) \subset C(A)$ converges to a function $f \in C(A)$.]

Theorem. (Weierstrass Approximation Theorem) $\forall f \in C(A)$ and $\forall \epsilon > 0$ there exists a polynomial function $p_\epsilon : A \to \mathbb{R}$ such that $d(f, p_\epsilon) < \epsilon$.

Corollary. The metric space $(C(A), d)$ is separable. [Hence, there is a countable dense subset of $C(A)$; namely, the collection of polynomials on A with rational (integer) coefficients.]

I.6. Axiom of Choice and its equivalents.

Some of the most controversial statements in mathematics are known as the Axiom of Choice, which is typically assumed as an axiom, and those statements proved by using it and are equivalent to it.

Axiom of Choice. If $\{X_\alpha\}_{\alpha\in\mathbb{A}}$ is a nonempty collection of nonempty sets, then $\Pi_{\alpha\in\mathbb{A}} X_\alpha$ is nonempty.

Let X be a set. Recall that a relation "$\leq$" on X is called a *partial ordering* if it is reflexive, antisymmetric and transitive. If $\leq$ also satisfies the property that

$$\text{if } x, y \in X, \text{ then either } x \leq y \text{ or } y \leq x$$

then it is called a *linear (or total) ordering.*

Let $(X, \leq)$ be a partially ordered set and $E \subset X$. An element $x \in X$ is called an *upper (lower) bound for* E if $y \leq x$ $(y \geq x)$ for all $y \in E$. An element $a \in X$ is called a *maximal (minimal)element of* X if $x \leq y$ $(y \leq x)$ then $x = y$. If every nonempty subset E of a partially ordered set $X, \leq)$ has a minimal element, then X is called a *well ordered* set and $\leq$ is called a *well ordering.*

Below are some of the statements equivalent to the Axiom of Choice.

Hausdorff Maximality Principle. Every partially ordered set has a maximal linearly ordered subset.

Zorn's Lemma. Let X be a partially ordered set. If every linearly ordered subset of X has an upper bound, then X has a maximal element.

Well Ordering Principle. Every nonempty set can be well ordered.