FUNDAMENTALS OF REAL ANALYSIS
by

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III. MEASURABLE FUNCTIONS AND LEBESGUE INTEGRAL

III.1. Measurable functions

Having the Lebesgue measure define, in this chapter, we will identify the collection of functions that are compatible with Lebesgue measure and then define Lebesgue integral on these functions.

Recall: f is continuous if and only if $f^{-1}(O)$ is open for all O open. Also, recall that

$$O = \bigcup_{k=1}^{\infty} I_k \Leftrightarrow f^{-1}(O) = \bigcup_{k=1}^{\infty} f^{-1}(I_k).$$

Continuous functions are well behaved for even Riemann-integral; hence we will naturally be compatible with measurable sets. We will consider a larger class of functions:

Definition. Let $f : E \to \mathbb{R}^{\#}$, where $E \in \mathcal{F}$. f is called **(Lebesgue) measurable** if for all $a \in \mathbb{R}$, the set $\{x \in E : f(x) > a\} \in \mathcal{F}$.

Fact.1 Let $f : E \to \mathbb{R}^{\#}$, $E \in \mathcal{F}$. The following are equivalent:

i) f is measurable.
ii) $\{x \in E : f(x) \geq a\} \in \mathcal{F}$ for all $a \in \mathbb{R}$.
iii) $\{x \in E : f(x) < a\} \in \mathcal{F}$ for all $a \in \mathbb{R}$.
iv) $\{x \in E : f(x) \leq a\} \in \mathcal{F}$ for all $a \in \mathbb{R}$.

Furthermore i)-iv) imply

v) $\{x \in E : f(x) = a\} \in \mathcal{F}$ for all $a \in \mathbb{R}^{\#}$.

Proof. (i) $\Rightarrow$ (ii): First, note that $\{x \in E : f(x) \geq a\} = f^{-1}([a, \infty])$. Also, since

$$[a, \infty] = \bigcap_n (a - \frac{1}{n}, \infty],$$

it follows that

$$\{x \in E : f(x) \geq a\} = f^{-1}\left(\bigcap_{n=1}^{\infty}(a - \frac{1}{n}, \infty]\right) = \bigcap_{n=1}^{\infty} f^{-1}\left((a - \frac{1}{n}, \infty]\right) \in \mathcal{F}.$$

(ii) $\Rightarrow$ (iii):

$$\{x \in E : f(x) < a\} = f^{-1}((-\infty, a)) = f^{-1}(\mathbb{R} \setminus [a, \infty]) = E \setminus f^{-1}([a, \infty]) \in \mathcal{F}.$$

(iii) $\Rightarrow$ (iv) and (iv) $\Rightarrow$ (i) are similar and left as an exercise.

For (v), if $a \in \mathbb{R}$, then $\{f(x) = a\} = f^{-1}(\{a\}) = f^{-1}((-\infty, a]) \cap f^{-1}([a, \infty]) \in \mathcal{F}$. If $a = \infty$,

$$\{f(x) = \infty\} = \bigcap_{n=1}^{\infty} \{f(x) > n\} \in \mathcal{F}. \quad \blacksquare$$

Examples. 1. Let $f : [0,1] \to \mathbb{R}$ be a function such that

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \\ 1 & \text{if } x \in \mathbb{Q}^c \end{cases}.$$

Then,

$$\begin{aligned} & f^{-1}(\tfrac{1}{2}, \infty) = \{x \in [0,1] : f(x) > \tfrac{1}{2}\} = \mathbb{Q}^c \cap [0,1] \in \mathcal{F}, \\ & \text{if } a = 1,\ f^{-1}(1,\infty) = \emptyset \in \mathcal{F}, \\ & \text{if } a = -1, \text{ then } f^{-1}(-1,\infty) = [0,1] \in \mathcal{F}. \end{aligned}$$

It follows that, for all $a \in \mathbb{R}$, $\{x \in [0,1] : f(x) > a\} \in \mathcal{F}$; hence, f is measurable.

2. Let $f(x)$ be the piecewise function defined as

$$f(x) = \begin{cases} 0 & \text{if } x \leq -1 \\ x^2 & \text{if } -1 < x < 1 \\ 2 & \text{if } x = 1 \\ 2 - x & \text{if } x > 1 \end{cases}$$

$$\begin{aligned} & \text{If } a < 0, \text{ then } f^{-1}(a,\infty) = \{x : f(x) > a\} = (-\infty, 2-a) \in \mathcal{F}. \\ & \text{If } a = 0, \text{ then } f^{-1}(0,\infty) = \{x : f(x) > 0\} = (-1,0) \cup (0,2) \in \mathcal{F}. \\ & \text{If } 0 < a < 1, \text{ then } f^{-1}(-1,\sqrt{a}) = (-1,-\sqrt{a}) \cup (\sqrt{a}, 2-a) \in \mathcal{F}. \\ & \text{If } 2 \leq a, \text{ then } f^{-1}(a,\infty) = \emptyset \in \mathcal{F}. \end{aligned}$$

Hence, f is measurable.

Fact.2 $f : E \to \mathbb{R}$, $E \in \mathcal{F}$, is measurable if and only if for all $O \in \mathcal{B}$, $f^{-1}(O) \in \mathcal{F}$.

Proof. Exercise.

Corollary.1 Continuous functions are measurable.

Proof. Exercise.

Corollary.2 Let $f : E \to \mathbb{R}$ be a measurable function on E.

i) If $D \subset E$, $D \in \mathcal{F}$, then $f|_D$ and $f|_{E \setminus D}$ are measurable on E.
ii) If $f = g$ almost everywhere on E, then g is measurable on E.

Proof. (i) Since $f : E \to \mathbb{R}$ be measurable on E,

$$\{x \in D : f(x) > a\} = (f|_D)^{-1}(a,\infty) = D \cap f^{-1}(a,\infty) \in \mathcal{F}$$

(ii) Let $A = \{x \in E : f(x) \neq g(x)\}$. Note that $m(A) = 0$; hence $A \in \mathcal{F}$. Then,

$$\begin{aligned} \{x \in E : g(x) > a\} &= \{x \in A : g(x) > a\} \cup \{x \in E \setminus A : g(x) > a\} \\ &= \underbrace{\{x \in A : g(x) > a\}}_{\in \mathcal{F}} \cup \underbrace{\{x \in E \setminus A : f(x) > a\}}_{(f|_{E \setminus A})^{-1}(a,\infty) \in \mathcal{F}} \in \mathcal{F}. \quad \blacksquare \end{aligned}$$

Fact.3 Let $E \in \mathcal{F}$ and $f, g : E \to \mathbb{R}$ be measurable functions and finite almost everywhere. Then:

i) $\alpha f + \beta g$ is measurable on E for all $\alpha, \beta \in \mathbb{R}$.
ii) f^2 is measurable on E (hence, fg is measurable on E).
iii) $\frac{1}{f}$ is measurable on E.

iv) $|f|$ is measurable on E.

Proof. (i) First, we show that αf is measurable on E for all $\alpha \in \mathbb{R}$. If $\alpha = 0$, then $\alpha f = 0$. Hence αf is measurable. If $\alpha > 0$, then

$$\{x \in E : (\alpha f)(x) > a\} = \left\{x \in E : f(x) > \frac{a}{\alpha}\right\} \in \mathcal{F}.$$

Similarly for $\alpha < 0$. Hence αf is measurable, and so is βg. Therefore, it is enough to show that $f + g$ is measurable on E. Notice that,

$$\{x \in E : f(x) + g(x) > a\} = \{x \in E : f(x) > a - g(x)\}$$

Now, there exists $r_x \in \mathbb{Q}$ such that $f(x) > r_x > a - g(x)$. Hence

$$\{x \in E : f(x) > a - g(x)\} = \bigcup_{r \in \mathbb{Q}} (\underbrace{\{x \in E : f(x) > r\}}_{\in \mathcal{F}} \cap \underbrace{\{x \in E : r > a - g(x)\}}_{\in \mathcal{F}}) \in \mathcal{F}.$$

(ii) First, observe that $fg = \frac{1}{2}[(f+g)^2 - f^2 - g^2]$. Therefore it is enough to show that f measurable implies f^2 is measurable. Now, we have

$$\begin{aligned} \{x \in E : (f^2)(x) > a\} &= \{x \in E : [f(x)]^2 > a\} \\ &= \underbrace{\{x \in E : f(x) > \sqrt{a}\}}_{\in \mathcal{F}} \cup \underbrace{\{x \in E : f(x) < -\sqrt{a}\}}_{\in \mathcal{F}} \in \mathcal{F}. \end{aligned}$$

(iii) If $a = 0$, then $\{x \in E : \frac{1}{f}(x) > a\} = \{x \in E : f(x) > 0\} \in \mathcal{F}$. So assume that $a > 0$. Then

$$\left\{x \in E : \frac{1}{f}(x) > a\right\} = \left\{x \in E : f(x) < \frac{1}{a}\right\} \in \mathcal{F}.$$

(iv) Follows from

$$\begin{aligned} \{x \in E : |f|(x) > a\} &= \{x \in E : |f(x)| > a\} \\ &= \{x \in E : f(x) > a\} \cup \{x \in E : f(x) < -a\} \in \mathcal{F}. \quad \blacksquare \end{aligned}$$

Definition. For functions $f, g : E \to \mathbb{R}$, define

$$\begin{aligned} (f \vee g)(x) &= \max\{f(x), g(x)\} \\ (f \wedge g)(x) &= \min\{f(x), g(x)\} \\ f^+(x) &= f(x) \vee 0 \\ -f^-(x) &= f(x) \wedge 0 \end{aligned}$$

Notice that, with the definitions above, we have that $|f| = f^+ + f^-$ and $f = f^+ - f^-$.

Corollary. If $f, g : E \to \mathbb{R}$, $E \in \mathcal{F}$ is measurable on E, then:

i) f^+, f^- are measurable on E.
ii) $(f \vee g)(x)$ and $(f \wedge g)(x)$ are measurable on E.

Proof. Exercise.

Convention. For the rest of this chapter $f : E \to \mathbb{R}$. E will always be measurable.

Definition. Let $A \subset \mathbb{R}$ be a set, then then the function

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}$$

is called the **characteristic function** of the set A.

Easy Fact. $E \in \mathcal{F}$ if and only if χ_E is a measurable function.

For, observe that,

$$\chi_E^{-1}(a, \infty) = \begin{cases} \emptyset & \text{if } a \geq 1 \\ E & \text{otherwise} \end{cases}$$

Hence χ_E is measurable (on E). If $E \notin \mathcal{F}$, $E \in \mathbb{R}$, then χ_E is not measurable.

Question: If f and g are measurable functions. What about $f \circ g$?

The answer is "No" in general.

Example. Recall that $f : [0, 1] \to [0, 2]$ defined by $f(x) = x + \varphi(x)$, where ϕ is the Cantor-Lebesgue function, is strictly increasing. Then, by Fact 9 in Chapter.II, there exists $A \in \mathcal{F}$ such that $f(A) \notin \mathcal{F}$.

Extend f continuously in a strictly increasing manner onto $\mathbb{R}$. Say $g : \mathbb{R} \to \mathbb{R}$ such that $g|_{[0,1]} = f$. Then g^{-1} is continuous. Next, consider χ_A and define $h(x) := (\chi_A \circ g^{-1})(x)$. Now

$$\begin{aligned} h^{-1}\left(\frac{1}{2}, \infty\right) &= (\chi_A \circ g^{-1})^{-1}\left(\frac{1}{2}, \infty\right) = (g \circ \chi_A^{-1})\left(\frac{1}{2}, \infty\right) \\ &= g\left(\chi_A^{-1}\left(\frac{1}{2}, \infty\right)\right) = g(A) = f(A) \notin \mathcal{F}. \end{aligned}$$

Theorem.1 Let f, g be measurable functions on their respective domains. If g is continuous, then $g \circ f$ is measurable.

Proof. Define $h(x) := (g \circ f)(x)$. If $O \subset \mathbb{R}$ is a Borel set, then we have $h^{-1}(O) = (g \circ f)^{-1}(O) = f^{-1}(g^{-1}(O))$. Since g is continuous, $g^{-1}(O) \in \mathcal{B}$. Hence $f^{-1}(g^{-1}(O)) \in \mathcal{F}$. ■

Recall that, for a given collection of functions $\{f_n\}$, $f : E \to \mathbb{R}$, we say that $f_n \to f$ **pointwise** if and only if for all $x \in E$, for all $\epsilon > 0$, there exists $N(\epsilon, x)$ such that whenever $n \geq N$ we have $|f_n(x) - f(x)| < \epsilon$.

Also, $f_n \to f$ **uniformly** if and only if for all $\epsilon > 0$, there exists $N(\epsilon)$ such that whenever $n \geq N$ we have $|f_n(x) - f(x)| < \epsilon$ for all $x \in E$.

Definition. Let $\{f_n\}$ be a sequence of real-valued functions on a measurable set E. We say that $\{f_n\}$ **converges almost everywhere (a.e.) to** $f : E \to \mathbb{R}$ if there exists $N \subset E$ with $m(N) = 0$ such that $f_n \to f$ pointwise on $E \setminus N$.

Remarks. Uniform convergence $\Rightarrow$ pointwise convergence $\Rightarrow$ almost everywhere convergence. Note that the converse is not always true (Exercise: Give examples).

Observation. Let $f_n \to f$ pointwise, where each f_n is measurable (on E). For $x \in E$, if $f(x) > a$, then $\exists m \geq 1$ such that $f(x) > a + \frac{1}{m}$. Thus, since $f_n \to f$ pointwise, for large enough n, we will have $f_n(x) > a + \frac{1}{m}$. This means that,

$$\exists m \; \exists N \text{ such that, } \forall n \geq N, \; f_n(x) > a + \frac{1}{m}. \quad (*)$$

Conversely, if (*) is satisfied, then $f(x) = \lim_n f_n(x) \geq a + \frac{1}{m}$. Hence, we have

$$\begin{aligned}\{x : f(x) > a\} &= \{x \in E : (\exists m \geq 1)(\exists N \geq 1)(\forall n \geq N)(f_n(x) > a + 1/m)\} \\ &= \bigcup_{m\geq 1}\left(\bigcup_{N\geq 1}\left(\bigcap_{n\geq N}\left\{x : f_n(x) > a + \frac{1}{m}\right\}\right)\right) \in \mathcal{F}.\end{aligned}$$

Theorem.2 Let $E \in \mathcal{F}$ and $f_n : E \to \mathbb{R}$ be a measurable function for all $n \geq 1$. If $f_n \to f$ almost everywhere on E, then f is measurable on E.

Proof. Since the set of $x \in E$ such that $\{f_n(x)\}$ fails to converge to $f(x)$ has measure zero, which is a measurable set, without loss of generality by restricting f_n's to a subset of E on which $f_n \to f$ pointwise, we can assume that $f_n \to f$ pointwise on E. Then by the observation above, for all $a \in \mathbb{R}$

$$\{x \in E : f(x) > a\} = \bigcup_{m\geq 1}\left(\bigcup_{N\geq 1}\left(\bigcap_{n\geq N}\left\{x : f_n(x) > a + \frac{1}{m}\right\}\right)\right) \in \mathcal{F}. \quad \blacksquare$$

Corollary. Let $E \in \mathcal{F}$, $\{f_n\}$ be a family of measurable functions on E. If $f_n \to f$ uniformly or pointwise on E, then f is measurable on E.

Definition. Let f_n be a sequence of real-valued measurable functions on $E \in \mathcal{F}$. Assume each f_n is bounded (i.e. there exits $M > 0$ such that $|f_n(x)| < M$ for all $x \in E$). Fix $x \in E$ and define:

$$\begin{aligned} f^*(x) &= \sup_{n\geq 1} f_n(x) \\ f_*(x) &= \inf_{n\geq 1} f_n(x) \\ \overline{f}(x) &= \limsup_{n\geq 1} f_n(x) \\ \underline{f}(x) &= \liminf_{n\geq 1} f_n(x) \end{aligned}$$

Then f^*, f_*, $\overline{f}$, and $\underline{f}$ are all well-defined real valued functions on E.

Exercise. Let f_n be a sequence of real-valued measurable functions on $E \in \mathcal{F}$.

1. Show that f^*, f_*, $\overline{f}$, and $\underline{f}$ are all measurable on E.
2. Show that $f_n \to f$ almost everywhere on E if and only if

$$m(\{x \in E : \limsup f_n(x) > \liminf f_n(x)\}) = 0.$$

Question: If $|f|$ is measurable, does it imply that f is measurable?

The answer is "NO" in general.

Exercise. Find an example of a non-measurable function f for which $|f|$ is measurable.

Definition. A function $f : E \to \mathbb{R}$ which is measurable and takes only finitely many values is called a **simple function**.

Fact.4 A function $f : E \to \mathbb{R}$ is simple if and only if there exists $a_1, a_2, \ldots, a_n \in \mathbb{R}$ and $E_1, E_2, \ldots E_n \in \mathcal{F}$ such that

$$E = \bigcup_{k=1}^{\infty} E_k \text{ and } f(x) = \sum_{k=1}^{n} a_k \chi_{E_k}(x).$$

Proof. ($\Leftarrow$) follows from the definition.
($\Rightarrow$) In order to show that f is measurable, assume $a_1, a_2, \ldots a_n$ are its values. Then we have $f^{-1}(\{a_k\}) = E_k \in \mathcal{F}$. If $i \neq j$, then $a_i \neq a_j$ and hence $E_i \cap E_j = \emptyset$, and $\bigcup_{k=1}^{n} E_k = E$. Let

$$f(x) = \sum_{k=1}^{n} a_k \chi_{E_k}(x).$$

It is left as an exercise to verify that both sides agree for all $x \in E$. ■

Note that the formula in the Fact.4 is a simple function in *canonical form.* Of course, the same function can be expressed in many different ways involving characteristic functions of different sets.

Fact.5 Let f and g be simple functions on $E \in \mathcal{F}$, then

i) $f + g$ is simple.
ii) fg is simple.
iii) $\frac{f}{g}$ is simple (provided $g \neq 0$).

Proof. Exercise.

Fact.6 Let $E \in \mathcal{F}$ and $f : E \to \mathbb{R}$ be a bounded measurable function. Then for all $\epsilon > 0$, there exists simple functions ϕ_ϵ and ψ_ϵ on E such that

$$\phi_\epsilon(x) \leq f(x) \leq \psi_\epsilon(x) \text{ on } E \text{ and } 0 \leq (\psi_\epsilon - \phi_\epsilon)(x) < \epsilon \text{ on } E.$$

Proof. Since f is bounded there exists $[a, b] \supset f(E)$. Partition $[a, b]$ into subintervals of length $\ell < \epsilon$, say

$$a < y_0 < y_1 < \cdots < y_{n-1} < y_n = b.$$

For $i = 0, 1, \ldots n - 1$ let $E_i = f^{-1}((y_i, y_{i+1}]) \in \mathcal{F}$. Then

$$\phi_\epsilon(x) = \sum_{i=0}^{n-1} y_i \chi_{E_i}(x) \leq f(x) \leq \sum_{i=0}^{n-1} y_{i+1} \chi_{E_i}(x) = \psi_\epsilon(x).$$

Also,

$$\psi_\epsilon(x) - \phi_\epsilon(x) = \sum_{i=0}^{n-1} (y_{i+1} - y_i) \chi_{E_i}(x) < \epsilon. \quad ■$$

Theorem.3 (Approximation by Simple Functions) Let $E \in \mathcal{F}$ and $f : E \to \mathbb{R}^{\#}$. Then f is measurable on E if and only if there exists a sequence $\{\phi_n\}$ of simple functions on E such that $\phi_n \to f$ almost everywhere and $|\phi_n| \leq f$ on E. Furthermore, if $f \geq 0$, then ϕ_n's can be chosen as $\phi_n \uparrow f$ almost everywhere on E.

Proof. Since $f = f^+ - f^-$ and $|f| = f^+ + f^-$ without loss of generality, we can assume f is non-negative (i.e. $f \geq 0$). Therefore let $f \geq 0$. Note that ($\Leftarrow$) part is Theorem.2 above.

($\Rightarrow$) Given $n \geq 1$, let $E_n = \{x \in E : f(x) \leq n\}$. Apply the previous fact with $\epsilon = \frac{1}{n}$. So there exists ϕ_n and ψ_n simple functions on E_n such that $\phi_n \leq f \leq \psi_n$ on E_n and $\psi_n - \phi_n < \frac{1}{n}$ on E_n. Now $\phi_n \leq f$ on E_n and $f - \phi_n < \frac{1}{n}$ on E_n.

Extend ϕ_n to the rest of E by letting $\phi_n(x) = n$ on $E \setminus E_n$. Now, we'll show that $\phi_n \to f$ pointwise on E.

If x is such that $f(x) < \infty$, then we have $f(x) < N$ for some N. Then, for any $n \geq N$, since $x \in E_n$ for some n, we have $0 \leq f(x) - \phi_n(x) \leq \frac{1}{n}$; hence, were done.

If x is such that $f(x) = \infty$, then take $\phi_n(x) = n$; it follows that

$$\phi_n(x) \to f(x)$$

pointwise. Hence $\phi_n \uparrow f$ pointwise. ■

Corollary. Let $E \in \mathcal{F}$, a function $f : E \to \mathbb{R}$ is bounded and measurable if and only if there exists $\{\phi_n\}$, simple functions on E, such that $\phi_n \to f$ uniformly.

Proof. Exercise.

Exercises

1. Let $E \in \mathcal{F}$ and $f : E \to \mathbb{R}$ be a function. f is a measurable function if and only if $f^{-1}(O) \in \mathcal{F}, \ \forall \ O \in \mathcal{B}$.

2. Let $\{f_n\}$ be a sequence of $\mathbb{R}$-valued measurable functions on $E \in \mathcal{F}$ and $f : E \to \mathbb{R}$ be measurable. $f_n \to f$ almost everywhere on E if and only if

$$m(\{x \in E : \limsup_n f_n(x) > \liminf_n f_n(x)\}) = 0.$$

3. Find a function $f : E \to \mathbb{R}, \ E \in \mathcal{F}$, such that $|f|$ is measurable whereas f is not.

III.2 Lebesgue Integration of Bounded Measurable Functions

Now that we sorted out those functions compatible with the (Lebesgue) measurable sets, we will proceed to define the Lebesgue integral (of measurable functions). Throughout we will assume that: $E \in \mathcal{F}$ with $m(E) < \infty$, all functions $f : E \to \mathbb{R}$ are bounded, and all simple functions will be in canonical form.

Definition. Let $\phi : E \to \mathbb{R}$ be a simple function defined as $\phi(x) = \sum_{i=1}^{n} a_i \chi_{E_i}(x)$. The **(Lebesgue) integral** of ϕ, denoted by $\int_E \phi(x)\ dm$, is defined as

$$\int_E \phi(x)\ dm = \sum_{i=1}^{n} a_i m(E_i).$$

<u>Notation:</u> $\int_E \phi(x)\ dm = \int_E \phi(x)\ dx = \int_E \phi\ dm = \int_E \phi$.

Remarks. 1. If we extend ϕ to $\mathbb{R}$ by letting $\phi(x) = 0$ for all $x \in \mathbb{R} \setminus E$, then

$$\int_E \phi(x)\ dm = \int_{\mathbb{R}} (\phi \cdot \chi_E)(x)\ dm.$$

2. The Lebesgue integral of a step function is the same as the Riemann integral of a step function.

Example. Let $E = [0,1]$ and let $\phi(x) = \chi_{\mathbb{Q}\cap[0,1]}$. Then

$$\int_E \phi(x)\,dx = 1 \cdot m(\mathbb{Q} \cap [0,1]) = 0.$$

Hence ϕ is Lebesgue integrable. Recall ϕ is not Riemann integrable!

Exercise. Calculate $\int_{[0,1]} \phi$, where $\phi(x) = \chi_{\mathcal{C}\cap\mathbb{Q}} + 2\chi_{\mathcal{C}\cap\mathbb{Q}^c} + 3\chi_{[0,1]\setminus\mathcal{C}}$.

Fact.7 Let $\phi, \psi : E \to \mathbb{R}$ be simple functions. Then

i) $\int_E (a\phi + b\psi) = a\int_E \phi + b\int_E \psi$ for all $a, b \in \mathbb{R}$.
ii) $\int_E \phi \leq \int_E \psi$ if $\phi \leq \psi$.
iii) $\int_E \phi = \int_{E_1} \phi + \int_{E_2} \phi$ where $E_1, E_2 \in \mathcal{F}$, $E = E_1 \cup E_2$, and $E_1 \cap E_2 = \emptyset$.

Proof. i) Let $\phi(x) = \sum_{i=1}^n a_i\chi_{E_i}(x)$ and $\psi(x) = \sum_{j=1}^m b_j\chi_{F_j}(x)$ be in canonical form. Let $\{A_k\}_{k=1}^N$ be defined as $A_k = E_i \cap F_j$ with an appropriate ordering. Then

$$\{A_k\}_{k=1}^N = \{E_i \cap F_j\}_{i,j=1}^{n,m}$$

is a disjoint collection with $\bigcup_{k=1}^N A_k = E$. Hence

$$a\phi + b\psi = \sum_{k=1}^N (aa_i + bb_j)\chi_{A_k}$$

where i, j corresponds to the set A_k with $A_k = E_i \cap F_j$. Therefore, we have

$$\begin{aligned}
\int_E (a\phi + b\psi) &= \sum_{k=1}^N (aa_i + bb_j)m(A_k) = \sum_{k=1}^N aa_i m(A_k) + \sum_{k=1}^N bb_j m(A_k) \\
&= a\sum_{k=1}^N a_i m(A_k) + b\sum_{k=1}^N b_j m(A_k) = a\sum_{i=1}^n a_i m(E_i) + b\sum_{j=1}^m b_j m(F_j) \\
&= a\int_E \phi + b\int_E \psi.
\end{aligned}$$

ii) Since $\phi \leq \psi$ on E, $\psi - \phi \geq 0$ on E. Also since $\psi - \phi$ is simple, by i) we have

$$\int_E (\psi - \phi) \geq 0 \Rightarrow \int_E \psi \geq \int_E \phi.$$

Proof of iii) is left as an exercise. ■

<u>Recall:</u> Given any $f : E \to \mathbb{R}$ bounded, we can always find (construct) simple functions ϕ and ψ such that $\phi \leq f \leq \psi$ on E and $0 \leq (\psi - \phi)(x)$ can be made arbitrarily small on E.

Definition. Let $E \in \mathcal{F}$ and $f : E \to \mathbb{R}$ be a bounded function. We say that the function f is **(Lebesgue) integrable over** E if

$$\sup\left\{\int_E \phi : \phi \leq f \text{ on } E\right\} = \inf\left\{\int_E \psi : \psi \geq f \text{ on } E\right\},$$

where ϕ and ψ are simple functions. The common value is called the **(Lebesgue) integral of f over E** and is denoted by

$$\int_E f(x)\,dx = \int_E f = \int_E f\,dm.$$

For the simplicity of the arguments below, we will denote:

$$\inf_{f\le\psi}\int_E \psi = U_f \text{ and } \sup_{f\ge\phi}\int_E \phi = L_f.$$

Theorem.4 Let $E \in \mathcal{F}$, $m(E) < \infty$, and $f : E \to \mathbb{R}$ be a bounded function. Then f is measurable on E if and only f is Lebesgue integrable over E

Proof. ($\Rightarrow$) Assume f is measurable and $|f| \le M$. Partition the interval $[-M, M]$ into $2n$ subintervals of equal length. Let

$$-M = y_{-n} < y_{-n+1} < \cdots < y_{n-1} < y_n = M$$

be the endpoints of these subintervals. Define $E_k = f^{-1}([y_k, y_{k+1}])$. Then, $E_k \in \mathcal{F}$ for all $k = -n, \ldots n$ with $E_k \cap E_\ell = \emptyset$ for $k \ne \ell$, and

$$\sum_{k=-n}^{n} m(E_k) = m(E).$$

Now define $\phi_n = \sum_{k=-n}^{n-1} y_k \chi_{E_k}$ and $\psi_n = \sum_{k=-n}^{n-1} y_{k+1}\chi_{E_k}$. Therefore, on the set E,

$$\phi_n(x) \le f(x) \le \psi_n(x),$$

where $\{\psi_n\}$ and $\{\phi_n\}$ are integrable. Then

$$\inf_{f\le\psi}\int_E \psi \le \int_E \psi_n \text{ and } \sup_{f\ge\phi}\int_E \phi \ge \int_E \phi_n.$$

Thus,

$$\inf_{f\le\psi}\int_E \psi - \sup_{f\ge\phi}\int_E \phi \le \int_E (\psi_n - \phi_n) = \int_E \left[\sum_{k=-n}^{n-1} (y_{k+1} - y_k)\chi_{E_k}\right] < \frac{M}{n} m(E).$$

Since n is arbitrary, we have $L_f = U_f$ and f is Lebesgue integrable.

($\Leftarrow$) Given that f is bounded on E and $L_f = U_f$, let $0 < \epsilon_n = \frac{1}{n}$ for $n \ge 1$. From these assumptions and the definitions of inf / sup, we can pick a sequence of simple functions $\{\psi_n\}$ and $\{\phi_n\}$ on E such that $\phi_n \le f \le \psi_n$ on E, and

$$0 \le \left(\int_E \psi_n - \int_E \phi_n\right) < \frac{1}{n}.$$

Let $\widetilde{\psi}_n = \inf_n \psi_n$ and $\widetilde{\phi}_n = \sup_n \phi_n$. Since each ψ_n and ϕ_n are measurable, we have that both $\widetilde{\psi}_n$ and $\widetilde{\phi}_n$ are measurable on E, and $\widetilde{\phi}_n \le f \le \widetilde{\psi}_n$ on E. We need to show $\widetilde{\psi}_n = \widetilde{\phi}_n$ almost everywhere on E. To show this let

$$A = \{x \in E : \widetilde{\phi}(x) < \widetilde{\psi}(x)\}.$$

We will show that $m(A) = 0$. To do this let $A_k = \{x \in E : \widetilde{\psi}(x) - \widetilde{\phi}(x) > 1/k\}$, then $\bigcup_{k\ge1} A_k = A$. Now let

$$B_k^n = \{x \in E : \psi_n(x) - \phi_n(x) > 1/k\}.$$

Then $A_k \subset B_k^n \in \mathcal{F}$ and, on E, $k(\psi_n - \phi_n) > 1$. So

$$m(B_k^n) = \int_E \chi_{B_k^n} \le \int_E k(\psi_n - \phi_n) = k\int_E (\psi_n - \phi_n) < \frac{k}{n}.$$

It follows that $m(A_k) = 0$. Hence, we have $m(A) = 0$. ■

Fact.8 Let $f : [a, b] \to \mathbb{R}$ be a bounded function. If f is Riemann integrable, then f is Lebesgue integrable with

$$\int_{[a,b]} f\, dm = \int_a^b f(x)\, dx.$$

Proof. Exercise.

Fact.9 Let $E \in \mathcal{F}$, $m(E) < \infty$, $f, g : E \to \mathbb{R}$ be bounded functions. Then we have:

a) $\int_E (\alpha f + \beta g) = \alpha \int_E f + \beta \int_E g$ for all $\alpha, \beta \in \mathbb{R}$.
b) If $f = g$ almost everywhere on E, then $\int_E f = \int_E g$.
c) If $f \leq g$ almost everywhere, then $\int_E f \leq \int_E g$ (Hence $|\int_E f| \leq \int_E |f|$).
d) If there exists $a, b \in \mathbb{R}$ such that $a \leq f(x) \leq b$ on E, then

$$am(E) \leq \int_E f \leq bm(E).$$

e) If $E_1, E_2 \in \mathcal{F}$, and $E = E_1 \cup E_2$ with $E_1 \cap E_2 = \emptyset$, then

$$\int_E f = \int_{E_1} f + \int_{E_2} f.$$

Proof. Exercise.

Exercise. Let $E \in \mathcal{F}$ with $m(E) < \infty$, and $\{f_n\}$ be a sequence of real valued, bounded measurable functions on E such that $f_n \to f$ uniformly. Then

$$\int_E f_n \to \int_E f.$$

The uniform convergence condition in the exercise above cannot be relaxed as the following example shows.

Example. Let $f_n : [0, 1] \to \mathbb{R},\ n \geq 1$, be given by

$$f_n(x) = \begin{cases} n^2 x \text{ if } x \in [0, \frac{1}{n}] \\ -n^2 x + 2n \text{ if } \ x \in [\frac{1}{n}, \frac{2}{n}] \\ 0 \ \text{ otherwise,} \end{cases}$$

and $f \equiv 0$ on $[0, 1]$. Then,

$$\int_{[0,1]} f_n = 1 \neq 0 = \int_{[0,1]} f\ !$$

Theorem.5 (Egoroff) Let $E \in \mathcal{F}$ with $m(E) < \infty$, and $\{f_n\}$ be a sequence of real valued measurable functions on E such that $f_n \to f$ almost everywhere. Then for all $\epsilon > 0$, there exists a measurable $A_\epsilon \subset E$, $m(E \setminus A_\epsilon) < \epsilon$, such that $f_n \to f$ uniformly on A_ϵ.

Proof. Given $\epsilon > 0$ and for $n, j \in \mathbb{Z}^+$, define

$$E_n^j = \bigcap_{k=n}^{\infty} \{x \in E : |f_k(x) - f(x)| < 1/j\}.$$

Then $E_n^j \in \mathcal{F}$ for all n, j. If $A \subset E$ such that $f_n \to f$ pointwise on A (i.e. $m(E \setminus A) = 0$), then for all $x \in A$, $x \in E_n^j$ for some n, j. Hence

$$A \subset \bigcup_{n=1}^{\infty} E_n^j.$$

Therefore

$$A \subset \bigcup_{n=1}^{\infty} E_n^j \subset E \Rightarrow m\left(\bigcup_{n=1}^{\infty} E_n^j\right) = m(E), \text{ since } f_n \to f \text{ a.e. (hence } \; m(A) = m(E)).$$

Observe for a fixed j, $E_n^j \subset E_{n+1}^j$. Hence by continuity of m, $m(E \setminus E_n^j) = m(E) - m(E_n^j)$ and

$$\lim_{n\to\infty} m(E \setminus E_n^j) = m(E) - \lim_{n\to\infty} m(E_n^j) = m(E) - m\left(\bigcup_{n=1}^{\infty} E_n^j\right) = 0.$$

Consequently, for all $j \geq 1$, we can find $n(j)$ such that $n \geq n(j)$ and $m(E \setminus E_n^j) < \frac{\epsilon}{2^j}$. Let $A_\epsilon = \bigcap_{j=1}^{\infty} E_{n(j)}^j$, then $A_\epsilon \in \mathcal{F}$. Then

$$m(E \setminus \bigcap_{j=1}^{\infty} E_{n(j)}^j) = m\left(\bigcup_{j=1}^{\infty}(E \setminus E_{n(j)}^j)\right) \leq \sum_{j=1}^{\infty} m(E \setminus E_{n(j)}^j) < \sum_j \frac{\epsilon}{2^j} = \epsilon.$$

Now, it is left as an exercise to show that $f_n \to f$ uniformly on A_ϵ. ■

Remark. By the example above, if $f_n \to f$ pointwise on $E \in \mathcal{F}$, in general, we cannot expect $\int_E f_n \to \int_E f$. However, under somewhat mild additional conditions we have a positive answer:

Theorem.6 (Bounded Convergence Theorem) Let $E \in \mathcal{F}$, $m(E) < \infty$, and $\{f_n\}$ be a sequence of uniformly bounded measurable, real valued functions on E. If $f_n \to f$ almost everywhere, then

$$\int_E f_n \to \int_E f.$$

Proof. Since $\{f_n\}$ is uniformly bounded, we can find a positive real number M such that $|f_n(x)| \leq M$. Then, by Egoroff's Theorem, given $\epsilon > 0$, there exists $A \subset E$ such that $m(E \setminus A) < \frac{\epsilon}{4M}$ and $f_n \to f$ uniformly on A. Then, find N large enough that, for $n \geq N$, $|f_n(x) - f(x)| < \frac{\epsilon}{2m(A)}$ for all $x \in A$. Now for such $n \geq N$,

$$\int_E f_n - \int_E f = \int_A (f_n - f) + \int_{E\setminus A} (f_n - f),$$

and hence,

$$|\int_E f_n - \int_E f| \leq \int_A |(f_n - f)| + \int_{E\setminus A} |(f_n - f)| \leq \int_A |f_n - f| + \int_{E\setminus A} 2M dm,$$

since $|f_n - f| \leq 2M$. Therefore, we have

$$\left|\int_E f_n - \int_E f\right| < \frac{\epsilon}{2m(A)} m(A) + \frac{2M\epsilon}{4M} = \epsilon. \quad ■$$

Remark. The BCT can be restated as saying that under the given conditions the "lim" and "$\int$" can be interchanged.

Examples. 1. Given

$$f(x) = \begin{cases} 1 & \text{if } x \in [1/2, 1] \cup [1/4, 1/3] \cup [1/6, 1/5] \cup \dots \\ 0 & \text{otherwise} \end{cases}$$

Define

$$\begin{aligned} f_1(x) &= \chi_{[1/2,1]} \\ f_2(x) &= \chi_{[1/4,1/3]} + \chi_{[1/2,1]} \\ &\vdots \\ f_n(x) &= \sum_{k=1}^{n} \chi_{[1/(2k),1/(2k+1)]} \\ &\vdots \end{aligned}$$

It is straightforward to show that $f_n \to f$ pointwise (hence, is left as an exercise). Then $\{f_n\}$ is a uniformly bounded sequence of measurable functions. By the Bounded Convergence Theorem,

$$\begin{aligned} \int_{[0,1]} f &= \lim_{n\to\infty} \int_{[0,1]} f_n = \lim_{n\to\infty} \left[\sum_{k=1}^{n} m\left(\left[\frac{1}{2k}, \frac{1}{2k-1} \right] \right) \right] \\ &= \lim_{n\to\infty} \left[\sum_{k=1}^{n} \left(\frac{1}{2k-1} - \frac{1}{2k} \right) \right] = \lim_{n\to\infty} \left[\sum_{k=1}^{2n} (-1)^{k+1} \frac{1}{k} \right] \\ &= \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = \ln(2). \end{aligned}$$

2. Let

$$f(x) = \begin{cases} 2x & \text{if } x \in [0,1] \cap \mathbb{Q} \\ 1-x & \text{if } x \in [0,1] \cap \mathbb{Q}^c \end{cases}$$

If we define $g(x) = 1 - x$ on $[0, 1]$, then $f = g$ almost everywhere and g is Riemann Integrable = Lebesgue Integrable. Hence,

$$\int_{[0,1]} f(x)\,dx = \int_{[0,1]\cap\mathbb{Q}} f(x)\,dx + \int_{[0,1]\cap\mathbb{Q}^c} f(x)\,dx = \int_0^1 g(x)\,dx = \frac{1}{2}.$$

Exercises

1. Using only the definition, evaluate the integrals: $\int_{[0,1]} \sqrt{x}\,dx$ and $\int_{[0,1]} x^{1/3}\,dx$.

2. Let C denote the Cantor Set. Find $\int_{[0,1]} f(x)\,dx$ if

$$f(x) = \begin{cases} x^2 & \text{if } x \in C \cap \mathbb{Q} \\ x & \text{if } x \in C \cap \mathbb{Q}^c \\ 1 & \text{if } x \in [0,1] \setminus C. \end{cases}$$

3. Prove that $f_n \to f$ uniformly on A_ϵ in Egoroff's Theorem.

III.3. Lebesgue Integral of Non-negative Functions

In the previous section we defined the Lebesgue integral for bounded measurable functions. Now, we will continue with not necessarily bounded functions. We will deal with this case in two steps: first we will define the Lebesgue integral of the functions $f : E \to \mathbb{R}^+$, and then extend it to more general functions $f : E \to \mathbb{R}$.

Definition. Let $E \in \mathcal{F}$ ($m(E) = \infty$ is allowed) and $f : E \to \mathbb{R}^+$ be measurable. We define

$$\int_E f dm = \sup\{\int_E hdm \ : \ h : E \to \mathbb{R} \text{ is bounded, measurable, } h \leq f,\ m(\{x : h(x) \neq 0\}) < \infty\}.$$

Remark.

(1) It is possible that $\int_E f = \infty$
(2) Functions $h : E \to \mathbb{R}$ such that $m(\{x \in E : \ h(x) \neq 0\}) < \infty$ are called functions with **finite support**. The set $\{x \in E : h(x) \neq 0\}$ is called the **support** of h.

Exercise. Find

$$\int_{[0,1]} \frac{1}{\sqrt{x}}\, dm.$$

Fact.10 Let $E \in \mathcal{F}$, $f, g : E \to E^+$ be measurable functons. Then

(a) $\int_E(\alpha f + \beta g) = \alpha \int_E f + \beta \int_E g$.
(b) If $f \leq g$ almost everywhere on E, then $\int_E f \leq \int_E g$.
(c) $\int_E f = \int_{E_1} f + \int_{E_2} f$ where $E = E_1 \cup E_2$ and $E_1 \cap E_2 = \emptyset$.

Proof. The result follows from the the same fact on bounded functions; hence, it is left as an exercise. ■

Theorem.7 (Fatou's Lemma) Let $E \in \mathcal{F}$ and $\{f_n\}$ be a sequence of $\mathbb{R}^+$-valued, measurable functions on E. If $f_n \to f$ almost everywhere on E, then

$$\int_E f\, dm \leq \liminf_{n\to\infty} \int_E f_n\, dm.$$

Proof. Let ϕ be a bounded, measurable, non-negative function with $m(\{\phi \neq 0\}) < \infty$ such that $\phi \leq f$ almost everywhere on E. For each n, define $h_n = f_n \wedge \phi = \min\{f_n, \phi\}$. Then:

i) h_n is bounded and measurable for all n.
ii) $h_n \leq f_n$.
iii) h_n's vanish outside $A = supp(\phi) = \{x : \ \phi(x) \neq 0\}$.
iv) $h_n \to \phi$ almost everywhere on A.

Then, by the Bounded Convergence Theorem,

$$\int_E \phi\ dm = \int_A \phi\ dm = \lim_{n\to\infty} \int_A h_n\ dm \leq \liminf_{n\to\infty} \int_E f_n\ dm.$$

(Last inequality follows since $\lim_n \int_E f_n\ dm$ may not exist.) Therefore,

$$\int_E f\ dm = \sup_{\phi\leq f} \int_E \phi\ dm \leq \liminf_{n\to\infty} \int_E f_n\ dm. \ ■$$

Remarks. 1. Strict inequality in Fatou's Lemma is possible.

2. If non-negativity of f_n's is removed, the conclusion of Fatou's Lemma is not valid anymore.

Exercise. Provide examples for each of the remarks made above.

Theorem.8 (Monotone Convergence Theorem) Let $E \in \mathcal{F}$, and $\{f_n\}$ be a sequence of $\mathbb{R}^+$-valued measurable functions on E such that $f_n \leq f_{n+1}$ a.e. (on E) for all $n \geq 1$. If $f_n \to f$ a.e., then

$$\lim_{n\to\infty} \int_E f_n \, dm = \int_E f \, dm.$$

Proof. From hypothesis, $f_n \leq f_{n+1}$ for all n almost everywhere on E. Hence

$$\int_E f_n \, dm \leq \int_E f \, dm$$

for all n. Therefore by Fatou's Lemma

$$\limsup \int_E f_n \, dm \leq \int_E f \, dm \leq \liminf \int_E f_n \, dm.$$

Hence, it follows that $\lim_{n\to\infty} \int_E f_n \, dm = \int_E f \, dm$. ■

Remarks.

1. MCT $\Rightarrow$ Fatou's Lemma.
2. MCT does not hold for Riemann Integral.
3. Monotonicity in MCT is essential.

Example. Let $E = \mathbb{R}\,([0,\infty])$ and $f_n(x) = \chi_{[n,\infty)}(x)$, $n \geq 1$. Then

$$f_n \to 0 \text{ pointwise (but not monotonically), whereas, } \infty = \int_{\mathbb{R}} f_n \not\to \ 0 = \int_{\mathbb{R}} f.$$

Corollary.1 Let $E \in \mathcal{F}$ and $\{u_n\}$ be a sequence of positive real valued functions on E. If

$$f = \sum_{n=1}^{\infty} u_n \text{ (pointwise), then } \int_E f \, dm = \sum_{n=1}^{\infty} \int_E u_n \, dm.$$

Proof. Define $f_n = \sum_{k=1}^{n} u_k$, for $n \geq 1$. Then $f_n \uparrow f$ a.e. on E. Apply MCT:

$$\lim_{n\to\infty} \int_E f_n \, dm = \int_E f \, dm \Rightarrow \lim_{n\to\infty} \sum_{k=1}^{n} \int_E u_k \, dm = \sum_{k=1}^{\infty} \int_E u_k \, dm = \int_E f \, dm. \quad ■$$

Corollary.2 Let $E \in \mathcal{F}$ and $\{E_n\} \subset \mathcal{F}$ such that $E = \bigcup_n E_n$ and $E_i \cap E_j = \emptyset$ if $i \neq j$. Then for any $f : E \to \mathbb{R}^+$ measurable,

$$\sum_{n=1}^{\infty} \int_{E_n} f \, dm = \int_E f \, dm$$

Proof. Let $u_n = f\chi_{E_n}$ and apply Corollary.1. ■

Theorem.9 (Chebychev's Inequality) Let $E \in \mathcal{F}$ and $f : E \to \mathbb{R}^+$ be a measurable function. Then for all $\lambda > 0$,

$$m(\{x \in E : f(x) > \lambda\}) \leq \frac{1}{\lambda} \int_E f \, dm.$$

Proof. Let $E_\lambda = \{x \in E : f(x) > \lambda\}$. Assume first that $m(E_\lambda) < \infty$ and let $\phi = \lambda\chi_{E_\lambda}$. Then on E_λ, $\phi \leq f$. Hence

$$\begin{array}{ccl} \displaystyle\int_E \phi\, dm & \leq & \displaystyle\int_E f\, dm \\ \| & & \\ \displaystyle\int_E \lambda\chi_{E_\lambda}\, dm & = & \lambda m(E_\lambda) \Rightarrow m(E_\lambda) \leq \dfrac{1}{\lambda}\displaystyle\int_E f\, dm. \end{array}$$

Next we will assume that $m(E_\lambda) = \infty$. If we let $E_\lambda^n = E_\lambda \cap [-n, n]$, then $E_\lambda^n \uparrow E_\lambda$ and $m(E_\lambda^n) < \infty$ for each n. Now, applying above case to E_λ^n and $\phi_n = \lambda\chi_{E_\lambda^n}$, we obtain

$$\lambda m(E_\lambda^n) \leq \int_E f\, dm.$$

Then, by continuity of m,

$$\lambda m(E_\lambda) = \lambda \lim_{n\to\infty} m(E_\lambda^n) \leq \int_E f\, dm.$$

Note that this gives equality on both sides since $m(E_\lambda) = \infty$. Hence,

$$m(E_\lambda) = \frac{1}{\lambda}\int_E f\, dm. \quad \blacksquare$$

Theorem.10 Let $E \in \mathcal{F}$ and $f : E \to \mathbb{R}^+$ be measurable. Then

$$\int_E f\, dm = 0 \Leftrightarrow f = 0 \text{ almost everywhere on } E.$$

Proof. ($\Rightarrow$) Assume $\int_E f\, dm = 0$ and consider, for any n, the set

$$S = \{x \in E : f(x) > 1/n\}.$$

Then by Chebychev's Inequality,

$$m(S) \leq n\int_E f\, dm = 0.$$

Now, since

$$\{x \in E : f(x) > 0\} = \bigcup_n \{x \in E : f(x) > 1/n\},$$

it follows that $f = 0$ a.e. on E.

($\Leftarrow$) Next, assume that $f = 0$ almost everywhere on E. Let, for all n, $E_n = E \cap [-n, n]$. On each E_n let h be a bounded, measurable, non-negative function such that $h \leq f$, and ϕ be any bounded simple non-negative function with $\phi \leq h$ (i.e. $0 \leq \phi \leq h \leq f$ a.e.). Hence ϕ must be 0 almost everywhere on E_n. Then

$$\int_{E_n} \phi = 0 \Rightarrow \int_{E_n} h = 0.$$

Since h is arbitrary, $\int_E f = 0$ follows from the definition. $\blacksquare$

Remark. Theorem.10 implies that $f = g$ a.e. on $E \in \mathcal{F} \iff \int_E f = \int_E g$.

Definition. Let $E \in \mathcal{F}$. A function $f : E \to \mathbb{R}^+$ is called **(Lebesgue) integrable** over E if

$$\int_E f\, dm < \infty.$$

Example. Let's show that the function $f(x) = \frac{1}{x^2}$ is Lebesgue-integrable on $[1, \infty)$ and calculate $\int_{[1,\infty)} \frac{1}{x^2}$. For, we will use MCT. First, let, for $n \geq 1$,

$$f_n(x) = \begin{cases} \dfrac{1}{x^2} \text{ if } 1 \leq x \leq n \\ 0 \text{ if otherwise} \end{cases}$$

Hence, each f_n is bounded and measurable on $[1, \infty)$. Thus,

$$\int_{[1,\infty)} f_n = \int_{[1,n]} f_n = \int_1^n \frac{1}{x^2}.$$

Since $\frac{1}{x^2}$ is Riemann-integrable on $[1, n]$, it is also Lebesgue integrable on $[1, n]$ (with both integrals being equal). Hence, $\int_{[1,\infty)} f_n = \int_1^n \frac{1}{x^2} = 1 - \frac{1}{n}$. Therefore, by MCT,

$$\int_{[1,\infty)} \frac{1}{x^2} = \lim_n \int_{[1,\infty)} f_n = 1.$$

Fact.11 Let $E \in \mathcal{F}$ and $f : E \to \mathbb{R}^+$ be integrable over E. Then $f(x) < \infty$ a.e. on E.

Proof. It is enough to show that $m(\{x \in E : f(x) = \infty\}) = 0$. For, first observe that $\{x \in E : f(x) = \infty\} = \bigcap_{n \geq 1}\{x \in E : f(x) > n\}$. Now, by Chebychev's Inequality,

$$m(\{x \in E : f(x) = \infty\}) \leq m(\{x \in E : f(x) > n\}) \leq \frac{1}{n} \int_E f.$$

Since $\int_E f < \infty$, letting $n \to \infty$, we obtain the desired result. ■

Theorem.11 (Beppo Levi's Lemma) Let $\{f_n\}$ be an increasing sequence of nonnegative measurable functions on $E \in \mathcal{F}$. If the sequence $\{\int_E f_n\}$ is bounded, then

i) $f_n \to^p f$ on E for some non-negative measurable function $f : E \to \mathbb{R}^\#$,
ii) $\lim_n \int_E f_n = \int_E f < \infty$, and
iii) f is finite a.e.

Proof. For each $x \in E$, the sequence $\{f_n(x)\}$ is an increasing sequence of real numbers; hence, it converges to an extended real number. Therefore, we can define f pointwise by

$$f(x) = \lim_n f_n(x), \quad \text{for all } x \in E.$$

Since $f_n \uparrow f$, by MCT, we have $\int_E f_n \to \int_E f$. Therefore, by the fact that $\{\int_E f_n\}$ is bounded, we have have $\int_E f < \infty$; and by Fact.11 above we must also have that f is finite a.e. ■

The following fact is another important property of integrable functions.

Theorem.12 Let $f : E \to \mathbb{R}^+$ be an integrable function on $E \in \mathcal{F}$. Then, for any $\epsilon > 0$ there exists $\delta > 0$ such that $\forall A \subset E,\ A \in \mathcal{F}$, with $m(A) < \delta$, we have $\int_A f < \epsilon$.

Proof. If $|f| < M$ for some M on E, take $\delta = \frac{\epsilon}{M}$. The the assertion follows.

If f is arbitrary integrable function, let

$$f_n(x) = \begin{cases} f(x) \text{ if } x \leq n \\ n \text{ if } \ x > n. \end{cases}$$

Then $\{f_n\}$ is a bounded sequence and $f_n \to^p f$. on E. So, by MCT, $\int_E f_n \to \int_E f$. Hence, given $\epsilon > 0$, there exists $N \geq 1$ such that if $n \geq N$, then $[\int_E f - \int_E f_n] < \frac{\epsilon}{2}$. In particular, if $n = N$, we have $\int_E (f - f_N) < \frac{\epsilon}{2}$. Now, pick $\delta < \frac{\epsilon}{2N}$, then for any $A \subset E$ with $m(A) < \delta$ we have

$$\int_A f = \int_A (f - f_N) + \int_A f_N \leq \int_E (f - f_N) + \int_A N < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \quad \blacksquare$$

Exercise. Let $f : \mathbb{R} \to \mathbb{R}^+$ be an integrable function. Define a set function $\nu : \mathcal{F} \to [0, \infty]$ by $\nu(A) = \int_A f dm, \quad A \in \mathcal{F}$. Show that

i) $0 \leq \nu(A) \leq \infty$ for all $A \in \mathcal{F}$.
ii) $\nu(A) \leq \nu(B)$ if $A \subset B,\ A, B \in \mathcal{F}$.
iii) $\nu(\emptyset) = 0 = \nu(\{a\})$ for any $a \in \mathbb{R}$.
iv) $\nu(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \nu(A_i)$ if $\{A_i\} \subset \mathcal{F}$ is a disjoint family of sets.
v) $\nu(A) = 0$ if $m(A) = 0$.

Question. For f and ν as in the exercise above, what can you say about

i) $\nu(I) = \ell(I)$ for any interval $I \subset \mathbb{R}$?
ii) $\nu(A + \alpha) = \nu(A)$ for any $A \in \mathcal{F}$ and $\alpha \in \mathbb{R}$?
iii) $m(A) = 0$ if $\nu(A) = 0$ for $A \in \mathcal{F}$?

III.4. Lebesgue Integral of Arbitrary Functions

In this section, having defined the (Lebesgue) integral of a non-negative measurable function, we will extend it to arbitrary measurable functions. First, recall that

(i) $f = f^+ - f^-$, and
(ii) if f is measurable, then so are f^+ and f^-.

Definition. A function $f : E \to \mathbb{R}$ where $E \in \mathcal{F}$, is called **(Lebesgue) integrable** over E if both f^+ and f^- are integrable over E. In that case,

$$\int_E f = \int_E f^+ - \int_E f^-.$$

Remark. Since f^+ and f^- are non-negative and integrable, the set on which they take ∞ as a value is of measure zero; hence, the set on which f may take $\infty + \infty$ or $\infty - \infty$ values is also of measure zero.

Fact.12 If $E \in \mathcal{F},\ f : E \to \mathbb{R}$ is integrable and $A \subset E$, $A \in \mathcal{F}$ with $m(A) = 0$, then

$$\int_E f = \int_{E \setminus A} f.$$

Proof. Exercise.

Fact.13 Let $E \in \mathcal{F},\ f : E \to \mathbb{R}$ be measurable and $g : E \to \mathbb{R}^+$ be integrable with $|f| \leq g$ on E. Then f is integrable with

$$\int_E f \leq \int_E g.$$

In particular, $|\int_E f| \leq \int_E |f|$.

Proof. Exercise.

Fact.14 If $E \in \mathcal{F}$ and $f, g : E \to \mathbb{R}$ are integrable over E, then

(a) αf is integrable over E and $\int_E \alpha f = \alpha \int_E f$.
(b) $f + g$ is integrable over E and $\int_E (f+g) = \int_E f + \int_E g$.
(c) $f \le g$ on E, then $\int_E f \le \int_E g$.
(d) If $E = E_1 \cup E_2$, where E_1 and E_2 are measurable and disjoint, then $f\chi_{E_1}$ and $f\chi_{E_2}$ are integrable with

$$\int_E f = \int_{E_1} f + \int_{E_2} f.$$

Proof. The proof of (a) and (c) are left as exercises. For (b) without loss of generality, if necessary, by removing a larger set of measure zero, we can assume that $f + g$ is finite on E. Thus

$$(f+g)^+ - (f+g)^- = f + g = (f^+ - f^-) + (g^+ - g^-) \text{ and}$$
$$(f+g)^+ + f^- + g^- = (f+g)^- + f^+ + g^+.$$

Since both sides are non-negative, we have

$$\int_E (f+g)^+ + \int_E f^- + \int_E g^- = \int_E (f+g)^- + \int_E f^+ + \int_E g^+.$$

If we reorganize we have

$$\int_E \left[(f+g)^+ - (f+g)^-\right] = \int_E (f^+ + f^-) + \int_E (g^+ - g^-).$$

Hence $f + g$ is integrable.

For the proof of (d), since f is integrable we know that $|f\chi_{E_1}| \le |f|$ and $|f\chi_{E_2}| \le |f|$. Now both $f\chi_{E_1}$ and $f\chi_{E_2}$ are integrable with $\chi_E = \chi_{E_1} + \chi_{E_2}$. ■

Theorem.13 (Dominated Convergence Theorem) Let $\{f_n\}$ be a sequence of real valued measurable functions on $E \in \mathcal{F}$. Let $g : E \to \mathbb{R}$ be an integrable function over E such that $|f_n| \le g$ for all n on E. If $f_n \to f$ almost everywhere on E, then f_n and f are integrable over E and

$$\int_E f_n \longrightarrow \int_E f.$$

Proof. Since g is integrable over E and $|f_n| \le g$, then each f_n is integrable over E. But then, since $f_n \to f$ almost everywhere, we also have $|f| \le g$ on E. Hence f is integrable over E.

Observe that $\{g - f_n\}$ is a sequence of non-negative integrable functions on E converging to $g - f$. Similarly $\{g + f_n\}$ is a sequence of non-negative functions converging to $g + f$. Now by applying Fatou's Lemma to $\{g - f_n\}$ we have

$$\int_E g - \int_E f = \int_E (g - f) \le \liminf \int_E (g - f_n) = \int_E g - \limsup \int_E f_n$$
$$\Rightarrow \int_E f \ge \limsup \int_E f_n.$$

Next, by applying Fatou's Lemma to $\{g + f_n\}$ we have

$$\int_E f + \int_E g = \int_E (g + f) \le \liminf \int_E (g + f_n) = \int_E g + \liminf \int_E f_n$$
$$\Rightarrow \int_E f \le \liminf \int_E f_n.$$

These two inequalities imply that $\int_E f_n \to \int_E f$. ■

Exercise. For each $n \geq 1$, let $f_n(x) = \frac{n^{3/2}x}{1+n^2x^2}$, $x \in [0,1]$. Show that

(a) $f_n \to 0$ almost everywhere on $[0,1]$.
(b) $|f_n(x)| \leq g(x)$ on $[0,1]$ for all n where $g(x) = \frac{1}{\sqrt{x}}$.
(c) $\lim_{n\to\infty} \int_{[0,1]} f_n = 0$.

Remark. Converse of the DCT is not valid!

Examples. Let $E = [0,1]$.

1. Let $f_n(x) = 1 - x$ for all n and $f(x) = x$. Then,
$$\int_E f_n \longrightarrow \int_E f, \text{ but } f_n \not\to f.$$
2. Let $f_n(x) = \frac{n-1}{n} - x$ and $f(x) = x$. Then
$$\int_E f_n \longrightarrow \int_E f, \text{ but } f_n \not\to f.$$
3. Define
$$\begin{aligned}
f_0 &= \chi_{[0,1]} \\
f_1 &= \chi_{[0,\frac{1}{2}]}, \quad f_2 = \chi_{[\frac{1}{2},1]} \\
f_3 &= \chi_{[0,\frac{1}{3}]}, \quad f_4 = \chi_{[\frac{1}{3},\frac{2}{3}]}, \quad f_5 = \chi_{[\frac{2}{3},1]} \\
f_6 &= \chi_{[0,\frac{1}{4}]}, \quad f_7 = \chi_{[\frac{1}{4},\frac{1}{2}]}, \quad f_8 = \chi_{[\frac{1}{2},\frac{3}{4}]}, \quad f_9 = \chi_{[\frac{3}{4},1]} \\
f_{10} &= \chi_{[0,\frac{1}{5}]}, \quad \ldots \\
&\vdots
\end{aligned}$$
Then we have $\int_E f_n \longrightarrow 0$, but $f_n \not\to 0$ almost everywhere.

Theorem.14 (Generalized Dominated Convergence Theorem) Let $\{f_n\}$ be a sequence of real valued measurable functions on $E \in \mathcal{F}$ such that $f_n \to f$ almost everywhere on E. Let $\{g_n\}$ be a sequence of real valued non-negative functions on E such that $g_n \to g$ almost everywhere on E. Also let $|f_n| \leq g_n$ for all n on E. If
$$\int_E g_n \longrightarrow \int_E g, \text{ then } \int_E f_n \longrightarrow \int_E f.$$

Proof. Exercise.

Corollary.1 Let $E \in \mathcal{F}$ and $f : E \to \mathbb{R}$ be integrable over E.

(a) If $E = \bigcup_{n=1}^{\infty} E_n$ (disjoint), $E_n \in \mathcal{F}$. Then
$$\int_E f = \sum_{n=1}^{\infty} \int_{E_n} f.$$

(b) Given $\{E_n\} \subset \mathcal{F},\ E_n \subset E$ and $E_n \uparrow$. If $A = \cup_n E_n$, then

$$\int_A f = \lim_{n\to\infty} \int_{E_n} f.$$

(c) Given $\{E_n\} \subset \mathcal{F}$, $E_n \subset E$, and $E_n \downarrow$. If $A = \cap_n E_n$, then

$$\int_A f = \lim_{n\to\infty} \int_{E_n} f.$$

Proof. Exercise. [Hint: Use Corollary.1 before Chebychev's Inequality.]

Corollary.2 Let $E \in \mathbb{R}$ and $f : E \to \mathbb{R}$ be integrable over E. Then given $\epsilon > 0$ there exists $\delta > 0$ such that for all $A \subset \mathcal{F}$, $A \subset E$ and $m(A) < \delta$, then

$$\int_A f < \epsilon.$$

Proof. Exercise.

The results stated above have some important and interesting ramifications. Let $\nu(A) = \int_A f$ for all $A \in \mathcal{F}$, where $f : \mathbb{R} \to \mathbb{R}$ is integrable. Then by Corollary.1(a) above,

$$\nu\left(\bigcup_{\substack{k=1\\ \text{disjoint}}}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \nu(A_k).$$

Also $\nu(\emptyset) = 0$, $0 \leq \nu(A)$, $A \subset B \Rightarrow \nu(A) \leq \nu(B)$. Notice that ν is <u>not</u> a measure; however, it acts like a measure. Such set functions are called **signed measures**.

It turns out that the converse of Corollary.2 is also valid if $m(E) < \infty$.

Fact.15 Let $E \in \mathcal{F}$, $m(E) < \infty$, and $f : E \to \mathbb{R}$ be a measurable function on E. If for all $\epsilon > 0$ there exists $\delta > 0$ such that for all $A \in \mathcal{F}$, $A \subset E$ and $m(A) < \delta$, we have that $\int_A f < \epsilon$, then f is integrable over E.

Proof. WLOG we can assume that $f \geq 0$. Take $\epsilon = 1$ and let δ_0 be the corresponding value such that whenever $A \subset E$, $A \in \mathcal{F}$, with $m(A) < \delta_0$, we have $\int_A f < 1$.

<u>Claim</u>: $E = \bigcup_{k=1}^{n} E_k$ *(disjoint) such that* $m(E_k) < \delta_0$ *for all* $1 \leq k \leq n$.

(The proof of this claim is left as an exercise.) Hence we now have

$$\int_A f\chi_{E_k} = \int_{E_k} f < 1 \text{ for all } \ 1 \leq k \leq n.$$

So each $f\chi_{E_k}$ is integrable. Therefore, $f = \sum_{k=1}^{n} f\chi_{E_k}$ is integrable. ■

Exercises

1. Let $f : [a, b] \to \mathbb{R}$ be a bounded function. If f is Riemann-integrable, then f is Lebesgue-integrable and

$$\int_{[a,b]} f(x)dm = \int_a^b f(x)dx.$$

2. Let $E \in \mathcal{F}$ with $m(E) < \infty$ and $\{f_n\}$ be a sequence of $\mathbb{R}$-valued, bounded measurable functions on E such that $f_n \to f$ uniformly on E. Then

$$\int_E f_n dm \to \int_E f dm.$$

[Do not use BCT!]

3. The function $f(x) = \frac{1}{\sqrt{x}}$ is measurable on $[0,1]$. Calculate $\int_{[0,1]} \frac{1}{\sqrt{x}} dm$. [Notice: f is not Riemann-integrable on $[0,1]$.]

4. Find $\int_{[0,1]} f(x)\, dm$ if

$$f(x) = \begin{cases} x^2 & \text{if } x \in C \cap \mathbb{Q} \\ x & \text{if } x \in C \cap \mathbb{Q}^c \\ 1 & \text{if } x \in [0,1] \setminus C \end{cases}$$

where C is the standard Cantor Set.

5. Let $\{f_n\}_{n \geq 1}$ be a sequence of $\mathbb{R}$-valued functions on $[0,1]$ defined as

$$f_n(x) = \begin{cases} 0 & \text{if } x \in (0, \frac{1}{n+1}) \cup [\frac{1}{n}, 1] \\ x^{-\frac{3}{2}} & \text{if } x \in [\frac{1}{n+1}, \frac{1}{n}). \end{cases}$$

Show that

a) $f_n \to 0$ a.e.
b) $\int_{[0,1]} f_n \to 0$.
c) There is no integrable function $g : [0,1] \to \mathbb{R}$ such that $f_n \leq g$ a.e. for all $n \geq 1$. How would you reconcile this result with the DCT?

III.5. Convergence in Measure

In regards to convergence of sequence of functions, in addition to uniform, pointwise and a.e. convergence, we can add another mode of convergence as follows.

Definition. Let $\{f_n\}$ be a sequence of real valued measurable functions on $E \in \mathcal{F}$ and let each f_n be finite almost everywhere on E. For $f : E \to \mathbb{R}$ measurable and finite almost everywhere on E, we say that the sequence $\{f_n\}$ **converge in measure** on E to f if for all $\alpha > 0$

$$\lim_{n\to\infty} m(\{x \in E : |f_n(x) - f(x)| \geq \alpha\}) = 0.$$

<u>Notation</u>: $f_n \to^m f$.

Remark. It is easy to see that if $f_n \to^u f$, then $f_n \to^m f$. Indeed, with an additional condition we can replace uniform convergence with a.e. convergence by the following result.

Fact.16 Let $m(E) < \infty$ and $\{f_n\}$ be a sequence of real valued measurable functions on $E \in \mathcal{F}$, and $f : E \to \mathbb{R}$ be a finite almost everywhere. Then

$$f_n \to^{a.e.} f \Rightarrow f_n \to^m f.$$

Proof. Clearly f is measurable on E (why?). Let $\alpha > 0$ and $\epsilon > 0$ be arbitrary. By Egoroff's Theorem, there exists $F \subset E$, $F \in \mathcal{F}$, such that $m(E \setminus F) < \epsilon$ and $f_n \to^u f$ on F. Hence there exists N such that when $n \geq N$ we have

$$|f_n(x) - f(x)| < \alpha$$

for all $x \in F$. Thus if $n \geq N$,

$$m(\{x \in E : |f_n(x) - f(x)| \geq \alpha\}) \leq m(E \setminus F) < \epsilon.$$

Hence $m(\{x \in E : |f_n(x) - f(x)| \geq \alpha\}) \to 0$. ■

Remarks. 1. The condition $m(E) < \infty$ is essential. (Exercise: Provide a counterexample if $m(E) = \infty$.)

2. The converse of Fact.16 is not true in general, either. (Exercise: provide an example). However, it's valid along a subsequence:

Theorem.15 If $f_n \to^m f$ on $E \in \mathcal{F}$, then there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $f_{n_k} \to^{a.e.} f$.

Proof. Let $\epsilon_k = \frac{1}{2^k}$. Applying the hypothesis, for each k there exists N_k such that when $n \geq N_k$, $m(\{x \in E : |f_n(x) - f(x)| \geq \frac{1}{k}\}) < \frac{1}{2^k}$. Let

$$E_k = \{x \in E : |f_{N_k}(x) - f(x)| > \frac{1}{k}\}.$$

Then

$$\sum_{k=1}^{\infty} m(E_k) < \infty.$$

Hence by Borel-Cantelli Lemma, for almost every x, the point x belongs to at most finitely many E_k's. That means for almost every x, there exists k_x such that $x \notin E_k$ for $k \geq k_x$. So, for almost every x,

$$|f_{n_k}(x) - f(x)| < \frac{1}{k}$$

for all $k \geq k_x$. Hence $f_{n_k} \to^p f$ almost everywhere on E. ■

Remark. In Fatou's Lemma, MCT and DCT, the condition $f_n \to^{a.e.} f$ on E can be replaced by $f_n \to^m f$ on E.

III.6. Riemann Integral vs Lebesgue Integral

Recall: Let $f : [a, b] \to \mathbb{R}$ be bounded, $\dot{\mathcal{P}}_n$ be a tagged part partition. Then

$$S(f, \dot{\mathcal{P}}_n) = \sum_{i=1}^{n} f(t_i)(x_{i+1} - x_i)$$

$$S(f, \dot{\mathcal{P}}_n) \longrightarrow \int_a^b f =: \text{Riemann Integral of } f \text{ on } [a, b].$$

Let's look at this process a little differently. For $f : [a, b] \to \mathbb{R}$ a bounded function and $\mathcal{P}_n$ a partition of $[a, b]$, define step functions $\{\alpha_n\}$ and $\{\beta_n\}$ by

$$\alpha_n(x) = \sum_{k=1}^{n} u_k \chi_{[x_k, x_{k+1}]}(x), \quad \beta_n(x) = \sum_{k=1}^{n} v_k \chi_{[x_k, x_{k+1}]}(x), \text{ respectively,}$$

where $u_k = \inf_{x \in [x_k, x_{k+1}]}\{f(x)\}$ and $v_k = \sup_{x \in [x_k, x_{k+1}]}\{f(x)\}$. Then,

(i) $\alpha_n \uparrow$ and $\beta_n \downarrow$ as $n \to \infty$ pointwise (or as $\|\mathcal{P}_n\| \to 0$).
(ii) For all $n \geq 1$, α_n and β_n are Lebesgue integrable (by construction).

(iii) For all $n \geq 1$, α_n and β_n are Riemann-integrable since,

$$L_n(f) = \int_a^b \alpha_n = \sum_{k=1}^{n} m_k(x_{k+1} - x_k) \text{ and}$$

$$U_n(f) = \int_a^b \beta_n = \sum_{k=1}^{n} v_k(x_{k+1} - x_k),$$

and that $L_n(f)$ and $U_n(f)$ are special cases of $S(f, \dot{\mathcal{P}}_n)$.

We also have that $L_n(f) \uparrow$ and $U_n(f) \downarrow$. Therefore, f is Riemann-integrable if and only if $\lim_n U_n = \lim_n L_n (= \int_a^b f)$.

In general, since f is bounded, both sequences α_n and β_n are uniformly bounded; hence, $\alpha_n(x) \to \alpha(x)$ and $\beta_n(x) \to \beta(x)$ pointwise, for some $\alpha : [a,b] \to \mathbb{R}$ and $\beta : [a,b] \to \mathbb{R}$. Now, the Dominated Convergence Theorem implies that

$$L_n(f) = \int_{[a,b]} \alpha_n \longrightarrow \int_{[a,b]} \alpha, \quad \text{where } \alpha = \lim_{n\to\infty} \alpha_n(x) \text{ and}$$

$$U_n(f) = \int_{[a,b]} \beta_n \longrightarrow \int_{[a,b]} \beta, \quad \text{where } \beta = \lim_{n\to\infty} \beta_n(x).$$

If $\int_{[a,b]} \alpha = \int_{[a,b]} \beta$, then the Lebesgue Integral of f exists and is equal to the common value. Therefore, any bounded Riemann-integrable function is Lebesgue integrable.

Fact.17 $f : [a,b] \to \mathbb{R}$ is continuous at $x \in [a,b]$ if and only if $\alpha(x) = \beta(x)$.

Proof. Exercise.

Theorem.16 Let $f : [a,b] \to \mathbb{R}$ be bounded.

(a) f is Riemann Integrable on $[a,b]$ if and only if f is almost everywhere continuous on $[a,b]$.
(b) If f is Riemann Integrable on $[a,b]$, then f is Lebesgue Integrable and

$$\int_{[a,b]} f = \int_a^b f.$$

Proof. From above observation, if (a) is proven, (b) follows easily.

(a) ($\Rightarrow$) Since f is Riemann Integrable, then

$$\lim_{k\to\infty} \int_a^b \alpha_k = \lim_{k\to\infty} \int_a^b \beta_k \Rightarrow \int_a^b \alpha = \int_a^b \beta.$$

Since $\alpha \leq \beta$, we have

$$\int_{[a,b]} (\beta - \alpha) = 0 = \int_a^b (\beta - \alpha) \Rightarrow \alpha = \beta \text{ almost everywhere.}$$

Hence by Fact.17, f is continuous almost everywhere on $[a,b]$.

($\Leftarrow$) If f is continuous almost everywhere, then by Fact.17 for almost every $x \in [a,b]$ $\alpha(x) = \beta(x) (= f(x))$. Since α and β are Lebesgue Integrable,

$$\int_{[a,b]} \alpha = \int_{[a,b]} \beta.$$

Hence we must have that if

$$L_n = \int_a^b \alpha_n = \int_{[a,b]} \alpha_n \text{ and } U_n = \int_a^b \beta_n = \int_{[a,b]} \beta_n,$$

then $|U_n - L_n| = 0$. That is f is Riemann Integrable and we must have

$$\int_a^b f = \lim_{n\to\infty} \int_a^b \alpha_n \left(= \lim_{n\to\infty} \int_a^b \beta_n \right). \quad \blacksquare$$

Important Remark. All the concepts introduced and developed in this chapter can be rephrased in an arbitrary σ-finite measure space $(X, \mathcal{A}, \mu)$ and for functions $f : X \to \mathbb{R}$ (with appropriate modifications). We will leave this as an exercise.