FUNDAMENTALS OF REAL ANALYSIS
by

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IV. DIFFERENTIATION AND SIGNED MEASURES

IV.1. Differentiation of Monotonic Functions

Question: Can we calculate $\int_E f$ easily? More explicitly, can we hope for a result like the Fundamental Theorem of Calculus for Riemann Integral?

Answer: Yes, but with lots of work and patience.

The following fact is well-known:

Proposition. Let $f : [a, b] \to \mathbb{R}$ be a monotonic function. Then the set of discontinuities of f is at most countable (hence has measure 0).

Exercise. Prove that if E be a countable subset of $[a, b]$, then there exists $f : (a, b) \to \mathbb{R}$ which is continuous only on $(a, b) \setminus E$.

Convention: Throughout this chapter, $[a, b]$ will be an interval with $-\infty < a < b < \infty$.

Definition. A collection $\mathfrak{F}$ of closed and bounded intervals is called a **Vitali cover** for a set $E \subset \mathbb{R}$ if for all $x \in E$, for all $\epsilon > 0$ there exists $I \in \mathfrak{F}$ such that $x \in I$ and $\ell(I) < \epsilon$.

Example. Let $E \in \mathbb{R}$ be a bounded set (say $E \subset [c, d]$). Then the collection $\mathfrak{F} = \{[\alpha, \beta] : c \leq \alpha < \beta \leq d, \alpha, \beta \in \mathbb{Q}\}$ is a Vitali Cover for E.

Next theorem is an important one!

Theorem.1 (Vitali Covering Lemma) Let $E \subset \mathbb{R}$ with $m^*(E) < \infty$ and let $\mathfrak{F}$ be a Vitali cover for E. Then for all $\epsilon > 0$ there exists a finite disjoint sub-collection $\{I_k\}_{k=1}^n \subset \mathfrak{F}$ such that

$$m^*\left(E \setminus \bigcup_{k=1}^n I_k\right) < \epsilon.$$

Comments:

(a) One can always find an open set O such that $E \subset O$ and $m(O) < \infty$.
(b) Without loss of generality, we can assume that

$$I \in \mathfrak{F} \Rightarrow I \subset O \Rightarrow m\left(\bigcup I_k\right) = \sum \underbrace{m(I_k)}_{=\ell(I_k)} \leq m(O) < \infty.$$

Hence $\ell(I_k) \to 0$ if the collection $\mathfrak{F}$ is infinite.
(c) For any interval I, $\widetilde{I}$ will be the closed interval having the same midpoint as I and has length 5 times the length of I.

Proof (of Theorem.1) We will consider two cases.

Case 1: *There exists a disjoint sub-collection $\{I_k\}_{k=1}^n \subset \mathfrak{F}$ such that for all $I \in \mathfrak{F}$, $I \cap (\bigcup_{k=1}^n I_k) \neq \emptyset$.*

In this case we claim that $E \subset \bigcup_{k=1}^{n} I_k$ (and hence the theorem is proven). If the claim is false, then there exists a point $x \in E \setminus \bigcup_{k=1}^{n} I_k$. Then $x \notin I_k$ for all $1 \leq k \leq n$. Since $\mathfrak{F}$ is a Vitali cover, there exists $I \in \mathfrak{F}$ such that $x \in I$. By assumption

$$I \cap I_{k_0} \neq \emptyset$$

for some $1 \leq k_0 \leq k$. Without loss of generality we can assume that $I \cap I_k = \emptyset$ for $k \neq k_0$. First of all $I \neq I_{k_0}$, hence $\ell(I \setminus I_{k_0}) > 0$. Let $\epsilon < \frac{1}{2}\ell(I \setminus I_k)$, then there exists $J \in \mathfrak{F}$ such that $x \in J$ and $\ell(J) < \epsilon$. Thus

$$J \cap \left(\bigcup_{k=1}^{n} I_k \right) = \emptyset.$$

This contradicts our assumption, proving the claim.

<u>Case 2</u>: *No such finite family exists. That is, given any finite disjoint collection* $\{I_k\}_{k=1}^{n}$, *there exists* $I \in \mathfrak{F}$ *such that* $I \cap (\bigcup_{k=1}^{n} I_k) = \emptyset$.

[In this case we will construct (inductively) a countably disjoint sub-collection $\{I_k\}_{k=1}^{\infty} \in \mathfrak{F}$ such that $E \setminus \bigcup_k I_k$ is as small as we wish.]

Let I_1 be arbitrary. By assumption there exists I (there may be others) such that $I \cap I_1 = \emptyset$. Pick $I_1 \in \mathfrak{F}$ such that $I_2 \cap I_1 = \emptyset$ and $\ell(I_2) > \frac{1}{2}S_1$ where $S_1 = \ell(I)$. Now assume we have a disjoint collection $\{I_k\}_{k=1}^{n} \subset \mathfrak{F}$ constructed in this manner. Then let

$$S_n = \sup\{\ell(I) : I \in \mathfrak{F},\, I \cap I_k = \emptyset,\, k = 1, \ldots n\}.$$

Pick $I_{n+1} \in \mathfrak{F}$ such that $I_{n+1} \cap I_k = \emptyset$ for all $k = 1, \ldots n$ and $\ell(I_{n+1}) > \frac{1}{2}S_n$. So $\{I_k\}_{k=1}^{n+1}$ is a disjoint family. Hence, by induction we have a countably disjoint family $\{I_k\}_{k=1}^{\infty} \subset \mathfrak{F}$ such that each $I_k \subset O$ for all k, where O is an open set such that $E \subset O$ and $m(O) < \infty$. Then $\ell(I_k) \to 0$ must hold.

<u>Claim</u>: *This collection has the property that for all* $n \geq 1$

$$E \setminus \bigcup_{k=1}^{n} I_k \subset \bigcup_{k=n+1}^{\infty} \widetilde{I}_k.$$

To prove this claim let $n \geq 1$ and let $x \in E \setminus \bigcup_{k=1}^{n} I_k$. Then there exists $I \in \mathfrak{F}$ such that $x \in I$ and

$$I \cap \left(\bigcup_{k=1}^{n} I_k \right) = \emptyset.$$

Then $I \cap I_m \neq \emptyset$ for some $m \geq n+1$ must hold (otherwise, if $I \cap I_m = \emptyset$ for all m, then $\ell(I_m) > \frac{1}{2}\ell(I)$ for all m, which implies that $\ell(E_m) \not\to 0$). So now we have $\ell(I_m) \geq \frac{1}{2}S_m$. Let c_m be the center of I_m. So

$$|x - c_m| < \frac{1}{2}\ell(I_m) + \ell(I) < \frac{1}{2}\ell(I_n) + 2\ell(I_m).$$

Hence x must belong to $\widetilde{I}_m$. Therefore $x \in \bigcup_{k=n+1}^{\infty} \widetilde{I}_k$. This implies

$$E \setminus \bigcup_{k=1}^{n} I_k \subset \bigcup_{k=n+1}^{\infty} \widetilde{I}_k.$$

This now proves the above claim.

So given $\epsilon > 0$, since $\{I_k\}$ are disjoint, $m(\bigcup_{k=1}^{\infty} I_k) = \sum_{k=1}^{\infty} \ell(I_k) < m(O)$. Hence, we can pick n such that $\sum_{k=n+1}^{\infty} \ell(I_k) < \frac{\epsilon}{5}$. Then

$$m^*\left(E \setminus \bigcup_{k=1}^{n} I_k\right) \leq m^*\left(\bigcup_{k=n+1}^{\infty} \widetilde{I}_k\right) \leq \sum_{k=n+1}^{\infty} \ell(\widetilde{I}_k) < \epsilon. \quad \blacksquare$$

<u>Observe</u>: if $f : [a, b] \to \mathbb{R}$ is continuous and differentiable on (a, b), by the Mean Value Theorem, if $f'(x) > \alpha > 0$ on $(c, d) \subset (a, b)$, then

$$f(d) - f(c) \geq \alpha(d - c). \qquad (*)$$

Exercise.1 Let $f : [a, b] \to \mathbb{R}$ be a continuous function which is differentiable on (a, b). Let $E = \{x \in (a, b) : f'(x) > \alpha\}$ and let

$$\mathfrak{F} = \{[c, d] \subset (a, b) : (*) \text{ holds}\}.$$

Then the collection $\mathfrak{F}$ is a Vitali Cover for E.

Definition. Let $f : [a, b] \to \mathbb{R}$. The **upper derivative** of f at $c \in (a, b)$ is the value (possibly $+\infty$) defined by

$$\overline{D}f(c) = \lim_{h\to 0} \left[\sup_{0<|t|\leq h} \frac{f(x + t) - f(x)}{t}\right].$$

Similarly, the **lower derivative** of f at $c \in (a, b)$ is

$$\underline{D}f(c) = \lim_{h\to 0} \left[\inf_{0<|t|\leq h} \frac{f(x + t) - f(x)}{t}\right].$$

If $\overline{D}f(c) = \underline{D}f(c)$ and is $< \infty$, then we say that f is **differentiable** at c, and call this common value as the **derivative of f** at c, which is denoted by $f'(c)$.

Exercise.2

(i) Let $f : [-1, 1] \to \mathbb{R}$ be given as

$$f(x) = \begin{cases} x & \text{if } 0 \leq x \in \mathbb{Q} \\ 0 & \text{if } 0 \leq x \in \mathbb{R} \setminus \mathbb{Q} \\ \frac{x}{2} & \text{if } x < 0. \end{cases}$$

Find $\overline{D}f(0)$ and $\underline{D}f(0)$.

(ii) Construct $f : [a, b] \to \mathbb{R}$ continuous such that $\overline{D}f(c) \neq \underline{D}f(c)$, for some $c \in (a, b)$.

(iii) Construct $f : [a, b] \to \mathbb{R}$ such that $\overline{D}f(c) = \infty$ whereas $\underline{D}f(c) = 0$, for some $c \in (a, b)$.

Fact.1 Let $f : [a, b] \to \mathbb{R}$ be increasing. Then for all $\alpha > 0$

$$(a)\ m^*(\{x \in (a, b) : \overline{D}f(x) \geq \alpha\}) \leq \frac{1}{\alpha}[f(b) - f(a)] \text{ and}$$

$$(b)\ m^*(\{x \in (a, b) : \underline{D}f(x) = +\infty\}) = 0.$$

Proof. (a) Let $\alpha, \epsilon > 0$ be arbitrary. Let

$$E_\alpha = \{x \in (a, b) : \overline{D}f(x) \geq \alpha\}.$$

Choose $0 < \beta < \alpha$, and let $\mathfrak{F} = \{[c, d] \subset (a, b) : f(d) - f(c) \geq \beta(d - c)\}$. Then $\mathfrak{F}$ is a Vitali Cover for E_α. Then by the Vitali Covering Lemma,

$$m^*(E_\alpha) = m^*\left(\left[E_\alpha \setminus \bigcup_{k=1}^{n} [c_k, d_k]\right] \cup \left[\bigcup_{k=1}^{n} [c_k, d_k]\right]\right) < \epsilon + m^*\left(\bigcup_{k=1}^{n} [c_k, d_k]\right)$$

for some disjoint sub-collection $\{[c_k, d_k]\}_{k=1}^n \subset \mathfrak{F}$. Hence

$$\begin{aligned} \epsilon + m^*\left(\bigcup_{k=1}^n [c_k, d_k]\right) &\leq \epsilon + \sum_{k=1}^n (d_k - c_k) < \epsilon + \frac{1}{\beta}\sum_{k=1}^n [f(d_k) - f(c_k)] \\ &\leq \epsilon + \frac{1}{\beta}[f(b) - f(a)] \Rightarrow m^*(E_\alpha) < \frac{1}{\alpha}[f(b) - f(a)]. \end{aligned}$$

For $E^\infty = \{x \in (a,b) : \overline{D}f(x) = +\infty\}$, $E^\infty \subset E_N$ for any $N \geq 1$. Therefore,

$$m^*(E^\infty) \leq m^*(E_N) \leq \frac{1}{N}[f(d) - f(c)] \longrightarrow 0 \text{ as } N \to \infty, \text{ implying (b).} \quad \blacksquare$$

Corollary. If f is monotonic and differentiable, then $f' < \infty$ almost everywhere.

Question: Which functions are differentiable? (In particular, are monotonic functions differentiable?)

Theorem.2 (Lebesgue Differentiation Theorem) Let $f : (a,b) \to \mathbb{R}$ be monotonic. Then f is differentiable almost everywhere on (a,b).

Proof. We will assume that f is increasing on (a,b). Let

$$E_{\alpha,\beta} = \{x \in (a,b) : \overline{D}f(x) > \beta > \alpha > \underline{D}f(x)\}.$$

Note that f fails to be differentiable on points at which $\underline{D}f(x) < \overline{D}f(x)$. Then the family $\{E_{\alpha,\beta}\}_{\alpha,\beta\in\mathbb{Q}}$ is a countable collection of sets on which f is not differentiable. Hence, if

$$E = \bigcup_{\substack{\alpha,\beta\in\mathbb{Q} \\ \alpha<\beta}} E_{\alpha,\beta},$$

we need to show that $m^*(E) = 0$, which will be the case if we show that $m^*(E_{\alpha,\beta}) = 0$ for each α and β. So, fix $\alpha < \beta$ and let $F = E_{\alpha,\beta}$. Given $\epsilon > 0$, pick O open such that $F \subset O \subset (a,b)$ and $m(O) < m^*(F) + \epsilon$. Let

$$\mathfrak{F} = \{[c,d] : f(d) - f(c) < \alpha(d - c)\}.$$

Then $\mathfrak{F}$ is a Vitali Cover for F by Exercise.1. Hence by the Vitali Covering Lemma there exists a finite collection $\{[c_k, d_k]\}_{k=1}^n \subset \mathfrak{F}$ such that

$$m^*\left(F \setminus \bigcup_{k=1}^n [c_k, d_k]\right) < \epsilon.$$

Then

$$\begin{aligned} \sum_{k=1}^n [f(d_k) - f(c_k)] &< \alpha \sum_{k=1}^n (d_k - c_k) = \alpha m\left(\bigcup_{k=1}^n [c_k, d_k]\right) \\ &\leq \alpha m(O) < \alpha(m^*(F) + \epsilon). \end{aligned}$$

Now focus on the interval (c_k, d_k). $f|_{(c_k,d_k)}$ is increasing. By Fact.1,

$$m^*(F \cap (c_k, d_k)) < \frac{1}{\beta}[f(d_k) - f(c_k)].$$

Thus

$$\beta \sum_{k=1}^n m^*(F \cap (c_k, d_k)) < \alpha(m^*(F) + \epsilon).$$

Now we have

$$\begin{aligned} m^*(F) &\leq m^*\left(F \cap \left(\bigcup_{k=1}^{n}[c_k, d_k]\right)\right) + m^*\left(F \setminus \bigcup_{k=1}^{n}[c_k, d_k]\right) \\ &\leq \sum_{k=1}^{n} m^*(F \cap (c_k, d_k)) + \epsilon \leq \frac{\alpha}{\beta}(m^*(F) + \epsilon). \end{aligned}$$

Hence

$$m^*(F) \leq \frac{\alpha}{\beta} m^*(F) + \frac{\alpha}{\beta}\epsilon \Rightarrow m^*(F) < \left(\frac{\alpha}{\beta}\epsilon\right)\left(\frac{1}{1-\frac{\alpha}{\beta}}\right) \Rightarrow m^*(F) < c\epsilon$$

where $c = \frac{\alpha}{\beta - \alpha}$. Since ϵ is arbitrary, it follows that $m^*(F) = 0$. ■

Corollary. Let $f : [a, b] \to \mathbb{R}$ be increasing. Then f' is integrable over $[a.b]$ with

$$\int_{[a,b]} f' \leq f(b) - f(a).$$

Remark. The inequality above can be strict.

Example. Let $f : [0, 1] \to \mathbb{R}$ be the Cantor-Lebesgue function. Then f is differentiable almost everywhere on $[0, 1]$ with $f' = 0$ almost everywhere. However, $f(1) - f(0) = 1$ and

$$\int_{[a,b]} f' = 0 \lneq f(1) - f(0).$$

Proof (of Corollary). Define, for $n \geq 1$, a sequence of functions $g_n : [a, b] \to \mathbb{R}$ by

$$g_n(x) = \frac{f(x + \frac{1}{n}) - f(x)}{\frac{1}{n}}.$$

Then each g_n is a measurable function. Furthermore, since f is differentiable almost everywhere, $g_n \to f'$ a.e. on $[a, b]$. Hence f' is measurable (we will assume that $f(x) = f(b)$ for $x \geq b$). Then by Fatou's Lemma

$$\int_{[a,b]} f' \leq \liminf \int_{[a,b]} g_n.$$

Now

$$\begin{aligned} \int_{[a,b]} g_n &= n\left[\int_{[a,b]} f(x + 1/n) - \int_{[a,b]} f(x)\right] = n\left[\int_{[a+\frac{1}{n}, b+\frac{1}{n}]} f(x) - \int_{[a,b]} f(x)\right] \\ &= n\left[\int_{[b,b+\frac{1}{n}]} f(x) - \int_{[a,a+\frac{1}{n}]} f(x)\right] = n\left[f(b)\frac{1}{n} - \int_{[a,a+\frac{1}{n}]} f(x)\right] \\ &\leq n\left[f(b)\frac{1}{n} - f(a)\frac{1}{n}\right] = f(b) - f(a), \text{ since } f(a) \leq f(x), \ \forall x \geq a. \quad ■ \end{aligned}$$

Remark. If $f : (a, b) \to \mathbb{R}$ is increasing, then, following the same line of proof as above, we have

$$\int_{(a,b)} f' \leq \sup_{x \in (a,b)} f(x) - \inf_{x \in (a,b)} f(x).$$

Exercises

1. Let $E \subset \mathbb{R}$ be a bounded set (say $E \subset [a,b]$). If $\mathcal{V} = \{[\alpha,\beta] : a \le \alpha < \beta \le b,\ \alpha,\beta \in \mathbb{Q}\}$, show that $\mathcal{V}$ is a Vitali cover for E.

2. Let $f : [a,b] \to \mathbb{R}$ be a continuous function which is differentiable on (a,b), and let $E = \{x \in (a,b) : f'(x) > \alpha\}$. If $\mathcal{V} = \{[c,d] \subset (a,b) : f(d) - f(c) \ge \alpha(d-c)\}$, show that $\mathcal{V}$ is a Vitali cover for E.

3. a) Find a function $f : [a,b] \to \mathbb{R}$ such that $\overline{D}f(c) = \infty$, and $\underline{D}f(c) = 0$ for some $c \in (a,b)$.

b) Determine $\overline{D}f(0)$ and $\underline{D}f(0)$ for

$$f(x) = \begin{cases} x\sin(\frac{1}{x}) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

c) Determine $\overline{D}f(x)$ and $\underline{D}f(x)$ for the function $f : [0,1] \to \mathbb{R}$ given by $f(x) = \chi_{\mathbb{Q}}(x)$.

4. Consider functions $f, g : [-1,1] \to \mathbb{R}$ given by

$$f(x) = \begin{cases} x^2\cos(\frac{1}{x^2}) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

$$g(x) = \begin{cases} x^2\cos(\frac{1}{x}) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Find $\overline{D}f(0),\ \underline{D}f(0),\ \overline{D}g(0)$ and $\underline{D}g(0)$.

IV.2. Functions of Bounded Variation and Absolute Continuity

In this section and the next, we will improve the assertion of the Corollary above. Since any monotone function is measurable, our aim is to generalize Lebesgue Differentiation Theorem (and hence the Corollary above) to largest possible class of measurable functions.

Definition. Let $f : [a,b] \to \mathbb{R}$ be a function. Let $\mathcal{P} = \{[x_i, x_{i+1}]\}_{i=1}^n$ be a partition of $[a,b]$. Then the quantity

$$V_a^b(f,\mathcal{P}) := \sum_{i=1}^n |f(x_i) - f(x_{i+1})|$$

is called as the **variation of f** over $[a,b]$ with respect to $\mathcal{P}$. The quantity

$$T_a^b(f) := \sup\{V_a^b(f,\mathcal{P}) : \mathcal{P} \text{ is a partition of } [a,b]\}$$

is called the **total variation of f**. If $T_a^b(f) < \infty$, then f is called a **function of bounded variation**.

Example. (1) Any increasing (decreasing) function is of bounded variation.

(2) *Lipschitz continuous functions* ($\exists c > 0$, such that $|f(x) - f(y)| \le c|x-y|,\ \forall x, \in [a,b]$) are of bounded variation.

(3) Let f be defined as

$$f(x) = \begin{cases} x\sin(\frac{1}{x}) & \text{if } x \in (0,1] \\ 0 & \text{if } x = 0. \end{cases}$$

Then f is <u>not</u> of bounded variation.

Fact.2 Let $f : [a,b] \to \mathbb{R}$. Then

(a) $V_a^b(f,\mathcal{P}) \le V_a^b(f,\mathcal{P}')$ where $\mathcal{P}'$ is a refinement of $\mathcal{P}$.
(b) For all $a \le c \le b$, $T_a^b(f) = T_a^c(f) + T_c^b(f)$.
(c) The function $x \mapsto T_a^x(f)$ is a non-decreasing function.
(d) The function $x \mapsto f(x) + T_a^x(f)$ is a non-decreasing function.

Proof. We will leave (a) and (b) as an exercise. (c) Assume $x < y$. Then we have $T_a^y(f) = T_a^x(f) + T_c^y(f)$ by (b). Thus, by the definition, $T_x^y(f) \ge 0$. Hence $T_a^y(f) \ge T_a^x(f)$.

(d) Assume that $x < y$. Then

$$\begin{aligned}[f(y) + T_a^y(f)] - [f(x) + T_a^x(f)] &= [f(y) - f(x)] + [T_a^y(f) - T_a^x(f)] \\ &= [f(y) - f(x)] + T_x^y(f).\end{aligned}$$

We know this is nonnegative since $T_x^y(f) \ge |f(y) - f(x)| = V_a^b(f,\mathcal{P})$ where $\mathcal{P} = \{[x,y]\}$. ■

Theorem.3 (Jordan Decomposition of a function) A function $f : [a,b] \to \mathbb{R}$ is of bounded variation if and only if $f = g - h$, where g and h are increasing on $[a,b]$.

Proof. ($\Rightarrow$) Write $f(x) = f(x) + T_a^x(f) - T_a^x(f)$. Then, letting $g = f(x) + T_a^x(f)$ and $h = T_a^x(f)$, the assertion follows from Fact.2(c) and (d).

($\Leftarrow$) Let $\mathcal{P} = \{a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b\}$ be a partition of $[a,b]$. Then

$$\begin{aligned}V_a^b(f,\mathcal{P}) &= \sum_{k=1}^n |f(x_k) - f(x_{k-1})| = \sum_{k=1}^n |[g(x_k) - g(x_{k-1})] - [h(x_k) - h(x_{k-1})]| \\ &\le \sum_{k=1}^n |g(x_k) - g(x_{k-1})| + \sum_{k=1}^n |h(x_k) - h(x_{k-1})| \\ &= [g(b) - g(a)] + [h(b) - h(a)] < \infty. \quad ■\end{aligned}$$

Corollary. Any function of bounded variation is differentiable almost everywhere (and f' is integrable).

Proof. By Theorem.3, write $f = g - h$, where $g, h : [a,b] \to \mathbb{R}$ are monotonic. Apply Lebesgue Differentiation Theorem. ■

Question: Is it true that

$$\int_{[a,b]} f' = f(b) - f(a)\,?$$

Note that the Cantor-Lebesgue function is of bounded variation by the previous theorem. Also $\varphi' = 0$ almost everywhere. Hence, the Cantor-Lebesgue function, although is of bounded variation, fails to satisfy the Fundamental Theorem of Calculus since

$$\varphi(1) - \varphi(0) = 1 \ne 0 = \int_{[0,1]} f'.$$

Definition. A function $f : [a,b] :\to \mathbb{R}$ is called **absolutely continuous** on $[a,b]$ if for all $\epsilon > 0$ there exists $\delta > 0$ such that for any finite disjoint collection $\{[a_k,b_k]\}_{k=1}^n$ of subintervals of $[a,b]$

$$\sum_{k=1}^n |b_k - a_k| < \delta \Rightarrow \sum_{k=1}^n |f(b_k) - f(a_k)| < \epsilon.$$

Remark. The Cantor-Lebesgue function is not absolutely continuous.

Examples. 1. Lipschitz continuous functions are absolutely continuous on any compact interval.

2. $\sqrt{x}$ is absolutely continuous on $[0,1]$.

Question: Are absolutely continuous functions differentiable?

Answer YES!!

Fact.3 if $f : [a,b] \to \mathbb{R}$ is absolutely continuous, then its of bounded variation (hence differentiable almost everywhere).

Proof. Let $f : [a,b] \to \mathbb{R}$ be absolutely continuous, let $\epsilon = 1$ and $\delta > 0$ be the appropriate real number satisfying the definition. Pick $N \in \mathbb{Z}^+$ such that $\frac{b-a}{N} < \delta$. Now let

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b \;\text{ be such that }\; x_k - x_{k-1} < \delta,\ 1 \le k \le n.$$

Then take $T_{x_k}^{x_{k+1}}(f) = \sup V_{x_k}^{x_{k+1}}(f, \mathcal{P})$. Now, for any partition $\mathcal{P}$ of $[x_k, x_{k+1}]$, say $\mathcal{P} = \{x_k = t_0 < t_1 < \cdots < t_{m-1} < t_m = x_{k+1}\}$,

$$V_{x_k}^{x_{k+1}}(f, \mathcal{P}) = \sum_{i=1}^{m} |f(t_i) - f(t_{i-1})|.$$

Since $\sum |t_i - t_{i-1}| \le |x_k - x_{k-1}| < \delta$ and since f is absolutely continuous,

$$\sum_{i=1}^{,} |f(t_i) - f(t_{i-1})| < 1 = \epsilon.$$

So $T_{x_k}^{x_{k+1}}(f) \le 1$. Therefore, $T_a^b(f) = \sum_{k=0}^{n-1} T_{x_k}^{x_{k+1}}(f) = n < \infty$. ■

Theorem.4 Let $f : [a,b] \to \mathbb{R}$ be absolutely continuous. If $f'(x) = 0$ almost everywhere on $[a,b]$, then f is constant.

Proof. It is enough to prove that $f(t) = f(a)$ for all $t \in [a,b]$. So given $\epsilon > 0$, let $\delta > 0$ be chosen to satisfy the absolute continuity of f. Let

$$E = \{x \in (a,t) : f'(x) = 0\}.$$

Then $E \in \mathcal{F}$ and $m(E) = t - a$. Given any $x \in E$, we can find $h_x > 0$ such that

$$|f(x + h_x) - f(x)| < \frac{\epsilon h_x}{t - a}$$

(since $f'(x) = 0$). Then the collection of intervals of the form $[x, x + h_x]$ forms a Vitali Cover. So, by Vitali Covering Lemma, there exists a finite disjoint sub-collection, say $\{[x_i, x_{i+1}]\}_{i=1}^n$ such that

$$m\left(E \setminus \bigcup_{i=1}^{n} [x_i, x_i + h_{x_i}]\right) < \delta.$$

Hence, letting $F = E \setminus \bigcup_{i=1}^n [x_i, x_i + h_i]$ and let $a = x_0$, $t = x_{n+1}$, and $h_0 = 0$, $h_i = h_{x_i}$, we have

$$F = [a, x_1] \cup [x_1 + h_1, x_2] \cup [x_2 + h_2, x_3] \cup \cdots \cup [x_{n-1} + h_{n-1}, x_n] \cup [x_n + h_n, t].$$

Since $m(F) < \delta$, by absolute continuity of f we have

$$\begin{array}{rcl} \sum_{k=0}^{n} |f(x_{k+1}) & - & f(x_k + h_k)| < \epsilon \\ & \| & \\ |f(x_1) & - & f(a)| + \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| + |f(t) - f(x_n + h_n)|. \end{array}$$

Then

$$\begin{aligned} |f(t) - f(a)| &= |f(a) + f(x_1) - f(x_1) + f(x_1 + h_1) - f(x_1 + h_1) - f(x_2) + f(x_2) + \dots \\ &\quad \dots - f(x_n) + f(x_n) - f(x_n + h_n) + f(x_n + h_n) - f(t)| \\ &\le |f(x_1) - f(a)| + \sum_{k=1}^{n} |f(x_k) - f(x_{k-1} + h_{k-1})| + |f(t) - f(x_n + h_n)| \\ &\quad + \sum_{k=1}^{n} |f(x_k + h_k) - f(x_k)| < \epsilon + \sum_{k=1}^{n} \frac{\epsilon h_k}{t - a} < 2\epsilon. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we have $f(t) = f(a)$ for all $t \in [a, b]$. ■

Exercises

1. Consider function

$$f(x) = \begin{cases} x \sin(\frac{1}{x}) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

a) Determine if f a function of bounded variation on $[0, 1]$.
b) Show that f uniformly continuous on on $[0, 1]$.

2. Consider functions $f, g : [-1, 1] \to \mathbb{R}$ given by

$$f(x) = \begin{cases} x^2 \cos(\frac{1}{x^2}) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

$$g(x) = \begin{cases} x^2 \cos(\frac{1}{x}) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Determine if f and g are of bounded variation on $[-1, 1]$.

3. Let $f, g : [0, 1] \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} x^2 \sin(\frac{1}{x}) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

$$g(x) = \begin{cases} x \sin(\frac{1}{x^2}) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Determine if f and g are absolutely continuous.

4. For any $E \subset \mathbb{R}$, let $BV(E)$ denote the collection of all $\mathbb{R}$-valued functions of bounded variation on E. Show that if $f, g \in BV(E)$ and $\alpha \in \mathbb{R}$, then $f+g,\ \alpha f \in BV(E)$. Is $fg \in BV(E)$? (Prove or provide a counterexample.)

5. Let $f \in BV([a,b])$. Show that there exists a countable family $\{\mathcal{P}_n\}_{n\geq 1}$ of partitions of $[a,b]$ such that

$$\lim_n V_a^b(f,\mathcal{P}_n) = T_a^b(f).$$

6. Show that for any Lebesgue integrable function $f : [a,b] \to \mathbb{R}$, the function $g(x) = \int_{[a,x)} f(y)dy$ is absolutely continuous.

7. Let $f, g : [0,1] \to \mathbb{R}$ be functions be given by

$$f(x) = \begin{cases} x^2|\sin(\frac{1}{x})| & \text{if } x > 0 \\ 0 & \text{if } x = 0, \end{cases} \qquad g(x) = \sqrt{x}.$$

Show that

a) f and g are absolutely continuous on $[0,1]$.
b) $f \circ g$ is absolutely continuous on $[0,1]$.
c) $g \circ f$ is not absolutely continuous on $[0,1]$.

IV.3. Fundamental Theorem for Lebesgue Integration

Definition. Let $f : [a,b] \to \mathbb{R}$ be integrable. The **indefinite integral of f** is defined as

$$F(x) = \int_{[a,x]} f + c, \ \ x \in [a,b].$$

Recall: If f is Riemann integrable with $F'(x) = f(x)$, then

$$\int_a^x f\,dx = [F(x) - F(a)].$$

Question: How nice is $F(x)$ if f is Lebesgue integrable?

Fact.4 Let $f : [a,b] \to \mathbb{R}$ be integrable. Then the indefinite integral F of f is uniformly continuous on $[a,b]$ and is of bounded variation (hence differentiable almost everywhere).

Proof. Observe if $y > x$, then $|F(y) - F(x)| \leq \int_{[x,y]} |f|$. Since f is integrable, given $\epsilon > 0$, there exists $\delta > 0$ such that whenever $A \in \mathcal{F}$ with $m(A) < \delta$, then

$$\int_A |f| < \epsilon.$$

So pick x and y such that $|x-y| < \delta$. Then $m([x,y]) < \delta$. So $\int_{[x,y]} |f| < \epsilon$ and F is uniformly continuous.

Let $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ be a partition of $[a,b]$, call it $\mathcal{P}$. Then

$$\sum_{i=1}^n |F(x_i) - F(x_{i-1})| \leq \sum_{i=1}^n \int_{[x_{i-1},x_i]} |f| = \int_{[a,b]} |f| < \infty.$$

So $V_a^b(f,\mathcal{P}) < \infty$. Therefore $T_a^b(F) < \infty$; and hence, F is of bounded variation. ■

Fact.5 Let $f : [a,b] \to \mathbb{R}$ be integrable. If $\int_{[a,x]} f = 0$ for all $x \in [a,b]$, then $f = 0$ almost everywhere.

Proof. Observe that the hypothesis states that $\int_E f = 0$ for any open or closed $E \subset [a,b]$. Assume that assertion is false. That is there exists $F \in \mathcal{F}$, $F \subset [a,b]$ such that $m(F) > 0$ and $f > 0$ (or $f < 0$) on F. Then there exists an open set $O \supset [a,b] \setminus F$ with $m(O) > b-a$. Then $[a,b] \setminus O$ is closed and $[a,b] \setminus O \subset F$ with $m([a,b] \setminus O) > 0$. Further $f > 0$ on $[a,b] \setminus O$. Hence $\int_{[a,b]\setminus O} f > 0$. This is a contradiction. ■

Theorem.5 Let $f : [a,b] \to \mathbb{R}$ be integrable and $F(x) = F(a) + \int_{[a,x]} f$. Then $F' = f$ almost everywhere on $[a,b]$.

Proof. Assume that f is bounded, say $|f| \leq M$. F is of bounded variation (Fact.4); and hence, differentiable a.e., implying that $F'(x)$ exists. Define $g_n : [a,b] \to \mathbb{R}$ by

$$g_n(x) = \frac{F(x+\frac{1}{n}) - F(x)}{\frac{1}{n}}.$$

Then

$$|g_n(x)| = n|F(x+1/n) - F(x)| = n\left|\int_{[x,x+\frac{1}{n}]} f\right| \leq n\int_{[x,x+\frac{1}{n}]} |f| \leq M.$$

Also $g_n(x) \to F'(x)$ almost everywhere. Hence the Bounded Convergence Theorem now implies

$$\lim_{n\to\infty} \int_{[a,b]} g_n = \int_{[a,b]} F' = F(b) - F(a) = \int_{[a,b]} f,$$

where the the last equality follows from the definition of F and the second equality follows from the fact that F is Riemann-integrable (since it is continuous by Fact.4). Therefore, $\int_{[a,b]}(F' - f) = 0$; and hence, by Fact.5, $F' = f$ almost everywhere.

When f is an arbitrary integrable function, do the same for $f_m = f \wedge M$, for any $m \in \mathbb{Z}^+$. ■

Theorem.6 (F.T.C. for Lebesgue Integral) Let $f : [a,b] \to \mathbb{R}$ be integrable. The function f is absolutely continuous on $[a,b]$ if and only if f' is integrable where

$$f(x) = f(a) + \int_{[a,x]} f' \quad \forall x \in [a,b].$$

Proof. ($\Leftarrow$) Since f' is integrable, by Corollary.2 of DCT, given $\epsilon > 0$, there exists $\delta > 0$ such that whenever $A \in \mathcal{F}$ with $m(A) < \delta$, we have $\int_A |f'| dm < \epsilon$. Let $\{[x_i, x_{i+1}]\}_{i=0}^n$ be a collection of sub-intervals of $[a,b]$. Then, from the assumption we have $f(y) - f(x) = \int_{[x,y]} f'$ for any $x, y \in [a,b]$; and hence,

$$\sum_{i=0}^n |f(x_{i+1}) - f(x_i)| = \sum_{i=0}^n \left|\int_{[x_i,x_{i+1}]} f'\right| \leq \sum_{i=0}^n \int_{[x_i,x_{i+1}]} |f'| \leq \int_E |f'|,$$

where $E = \bigcup_{i=0}^n [x_i, x_{i+1}]$. Therefore, for any collection $\{[x_i, x_{i+1}]\}_{i=0}^n$ such that

$$\sum_{i=0}^n |x_{i+1} - x_i| < \delta,$$

we have

$$\sum_{i=0}^n |f(x_{i+1}) - f(x_i)| \leq \int_E |f'| < \epsilon.$$

Hence, f must be absolutely continuous.

(⇒) Assume that f is absolutely continuous. Then, by the Corollary of Theorem.3, f is differentiable and f' is integrable. So let

$$g(x) = \int_{[a,x]} f'.$$

Then $f' = g'$ almost everywhere by Theorem.5. Now look at the function $h := f - g$. Note that h is absolutely continuous and $h' = 0$ almost everywhere. Thus h must be constant by Theorem.4. So

$$f(x) = h(x) + g(x) = h(a) + g(x) = h(a) + \int_{[a,x]} f' = f(a) + \int_{[a,x]} f'. \quad \blacksquare$$

Corollary.1 (Lebesgue Decomposition of a function) Let $f : [a,b] \to \mathbb{R}$ be of bounded variation. Then there exists functions of bounded variation $g, h : [a,b] \to \mathbb{R}$ such that g is absolutely continuous on $[a,b]$, $h' = 0$ almost everywhere on $[a,b]$ and $f = g - h$ almost everywhere.

Proof. Set $g(x) = \int_{[a,x]} f'$ and $h = f - g$. ■

Corollary.2 Every function of bounded variation is the indefinite integral of its derivative.

Proof. Exercise.

Question. Can we claim that FTC holds in the setting of Corollary.1 and Corollary.2?

Exercise. Show that $f : [a,b] \to \mathbb{R}$ is absolutely continuous if and only if $\int_{[a,b]} |f'| = T_a^b(f)$.

IV.4 Signed Measures and Radon-Nikodým Theorem

We will let $(X, \mathcal{A})$ be a measurable space and $(X, \mathcal{A}, \mu)$ be a measure space. Recall that if $f : X \to \mathbb{R}^+$ is $(\mu-)$ measurable and $f \geq 0$, then the set function

$$\nu = \int_E f \, d\mu,$$

$E \in \mathcal{A}$, defines a measure on $\mathcal{A}$.

Question: What if non-negativity of f is dropped? Then $f = f^+ - f^-$ where f^+ and f^- are non-negative. So

$$\nu(E) = \int_E f \, d\mu = \int_E f^+ \, d\mu - \int_E f^- \, d\mu.$$

Note that now ν is not a measure anymore; however, it is still a nice set function since:

- $\nu(\emptyset) = 0$
- $\nu(E)$ is well defined (as an extended real number).
- If $\{E_n\}_n \subset \mathcal{A}$ is a disjoint collection, then

$$\begin{aligned} \nu\left(\bigcup_{n=1}^{\infty} E_n\right) &= \int_{\bigcup_n E_n} f = \int_{\bigcup_n E_n} f^+ - \int_{\bigcup_n E_n} f^- \\ &= \sum_{n=1}^{\infty} \int_{E_n} f^+ - \sum_{n=1}^{\infty} \int_{E_n} f^- = \sum_{n=1}^{\infty} \int_{E_n} f = \sum_{n=1}^{\infty} \nu(E_n). \end{aligned}$$

Definition. Let $(X, \mathcal{A})$ be a measurable space. A set function $\nu : \mathcal{A} \to \mathbb{R}^{\#}$ such that

- $\nu(\emptyset) = 0$
- ν takes at most one of either $+\infty$ or $-\infty$ as a value.
- If $\{E_n\} \subset \mathcal{A}$ is a disjoint collection of measurable sets, then

$$\nu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \nu(E_n)$$

  and the series is absolutely convergent if $\nu(\bigcup_n E_n) < \infty$,

is called a **signed measure** (on $\mathcal{A}$ or X).

Question: Can we characterize signed measures in terms of simple (ordinary) measures?

Definition. Let $(X, \mathcal{A})$ be a measurable space and ν a signed measure on $\mathcal{A}$. A set $A \in \mathcal{A}$ is called **ν-positive (ν-negative, ν-null)** if for every measurable $E \subset A$, $\nu(E) \geq 0$ ($\nu(E) \leq 0$, $\nu(E) = 0$, resp.).

Remark. (1) If $A \subset B$ and $|\nu(B)| < \infty$, then $|\nu(A)| < \infty$.

(2) Every ν-null set has ν-measure zero; however a set with ν-measure zero may not be ν-null. (Exercise: Find an example of such a set.)

Fact.6 Let ν be a signed measure on $(X, \mathcal{A})$. Then

(a) Any measurable subset of a ν-positive set is ν-positive.
(b) Any countable union of ν-positive sets is a ν-positive set.

Proof. Exercise.

Fact.7 (Continuity of signed measures) Let ν be a signed measure on $(X, \mathcal{A})$ and $\{A_k\}_k \subset \mathcal{A}$ be a collection of subsets of X.

(a) If $A_k \subset A_{k+1}$ for all $k \geq 1$, then $\nu(\cup_1^\infty A_k) = \lim_k \nu(A_k)$.
(b) If $A_k \supset A_{k+1}$ for all $k \geq 1$, then $\nu(\cap_1^\infty A_k) = \lim_k \nu(A_k)$.

Proof. Exercise. [Hint: Mimic the proof of Theorem.3 in Chapter II.)

Next, we will show that given a measurable space $(X, \mathcal{A})$ and a signed measure on it, one can decompose X into a disjoint union of a positive and a negative set w.r.t. the signed measure. For, we need the following statement.

Lemma. If ν is a signed measure on $(X, \mathcal{A})$ and $E \in \mathcal{A}$ such that $0 < \nu(E) < \infty$, then there exists a measurable set $A \subset E$ which is positive with $\nu(A) > 0$.

Proof. See p: 344 of Royden.

Theorem.7 (Hahn Decomposition) If ν is a signed measure on $(X, \mathcal{A})$, then there exist a positive set P and a negative set N such that $P \cup N = X$ and $P \cap N = \emptyset$.

Proof. By the definition of signed measure we can assume that ν does not assume $+\infty$ (for otherwise, consider $-\nu$). Define

$$\lambda = \sup\{\nu(E) : E \text{ is a positive set}\}.$$

Since $\emptyset$ is a positive set, $\lambda \geq 0$. Let $\{A_k\}$ be a countable collection of positive sets such that $\lambda = \lim_k \nu(A_k)$. If $P = \cup_k A_k$, then P is a positive set with $\nu(P) = \lambda$. Since ν does not assume $+\infty$, it d follows that $\lambda < \infty$.

Next, let $N = X \setminus P$; we will prove that N is a negative set. For, assume that N is not a negative set. Hence, $\exists B \subset N$ with positive measure. Then, by Lemma above, there is a positive set $A \subset B$ with positive measure. It follows that $P \cup A$ is a positive set and $\nu(P \cup A) = \nu(P) + \nu(A) > \lambda$, which is a contradiction. ∎

Remark. Hahn decomposition is not unique; however, it is unique up to a set of measure zero.

Observe that, given a measure space $(X, \mathcal{A}, \mu)$, one can find functions $f, g : X \to \mathbb{R}$ with $A = supp(f)$, $B = supp(g)$, where $A \cap B = \emptyset$, and $A \cup B = X$. Hence, if $\nu_1(.) = \int f d\mu$ and $\nu_2(.) = \int g d\mu$, then A is ν_2-null and B is ν_1-null; i.e., ν_1 and ν_2 live on disjoint sets. More generally,

Definition. Let $(X, \mathcal{A})$ be a measurable space. The signed measures ν_1 and ν_2 on $\mathcal{A}$ are called **singular** (or **ν_1 (ν_2) is singular with respect to ν_2 (ν_1)**) if there exists $E_1,\ E_2 \in \mathcal{A}$ such that $E_1 \cap E_2 = \emptyset,\ E_1 \cup E_2 = X$ and E_i is ν_i-null, $i = 1, 2$.

Notation. $\nu_1 \perp \nu_2$.

Theorem.8 (Jordan Decomposition of a measure) If ν is a signed measure on a measurable space $(X, \mathcal{A})$, then there exist unique positive measures ν^+ and ν^-such that $\nu = \nu^+ - \nu^-$ and $\nu^+ \perp \nu^-$.

Proof. The existence follows from Hahn decomposition by letting $\nu^+(E) = \nu(E \cap P)$ and $\nu^-(E) = -\nu(E \cap N \cap)$ for any $E \in \mathcal{A}$. Then $\nu = \nu^+ - \nu^-$ and $\nu^+ \perp \nu^-$.

For uniqueness, if $\nu = \mu^+ - \mu^-$ and $\mu^+ \perp \mu^-$ is another decomposition, then pick $A, B \in \mathcal{A}$ with $A \cap B = \emptyset,\ A \cup B = X$ and $\mu^+(B) = 0,\ \mu^-(A) = 0$. Then this gives another Hahn decomposition for ν, and consequently, $P \triangle A$ will be ν-null. Hence, for any $E \in \mathcal{A}$,

$$\mu^+(E) = \mu^+(E \cap A) = \nu(E \cap A) = \nu(E \cap P) = \nu^+(E),$$

i.e., $\mu^+ = \nu^+$. Similarly, $\mu^- = \nu^-$. ∎

Observe the similarity between the decomposition in Theorem.8 and the Jordan decomposition of a function of bounded variation (into difference of two monotonic functions). Hence, the following definition is appropriate.

Definition. The measures ν^+ and ν^- are called the **positive** and **negative variations of** ν; $\nu = \nu^+ - \nu^-$ is called the **Jordan decomposition of** ν, and for the measure $|\nu|$ defined by $|\nu| = \nu^+ + \nu^-$, $\|\nu\| := |\nu|(X)$ is called the **total variation of** ν.

Example. Let $f : \mathbb{R} \to \mathbb{R}$ be Lebesgue integrable function, and let $\nu(E) = \int_E f dm$ for any $E \in \mathcal{F}$. Then ν is a signed measure on $(\mathbb{R}, \mathcal{F})$. Let $P = \{x \in \mathbb{R} : f(x) \geq 0\}$, $N = \{x \in \mathbb{R} : f(x) < 0\}$ and define, for $E \in \mathcal{F}$,

$$\nu^+(E) = \int_{P \cap E} f dm \ \text{ and } \ \nu^-(E) = -\int_{N \cap E} f dm.$$

Then $\{P, N\}$ is a Hahn decomposition of $\mathbb{R}$ w.r.t. ν, and $\nu = \nu^+ - \nu^-$ is Jordan decomposition of ν.

Exercises. 1) $|\nu|(X) = \sup\{\sum_{k=1}^n |\nu(E_k)| : \{E_k\}_1^n \subset \mathcal{A}$ disjoint collection$\}$.

2) $E \in \mathcal{A}$ is ν-null if and only if $|\nu|(E) = 0$.

3) $\nu \perp \mu$ if and only if $|\nu| \perp \mu$ if and only if $\nu^+ \perp \mu$ and $\nu^- \perp \mu$.

4) If ν omits the value ∞, then $\nu^+(X) = \nu(P)$; hence ν^+ is a finite measure and ν is bounded above by $\nu^+(X)$. The same is also valid if ν omits the value $-\infty$ (with appropriate changes).

Let $f = \chi_P - \chi_N$, where $X = P \cup N$ is the Hahn decomposition for ν, and let $\mu = |\nu|$, then $\nu(E) = \int_E f d\mu$ for any $E \in \mathcal{A}$. Generalizing this idea, one defines integration w.r.t. a signed measure ν by

$$\int f d\nu = \int f d\nu^+ - \int f d\nu^-$$

for any ν^+ and ν^--integrable function f.

In regards to differentiation w.r.t. a signed measure, we will follow similar steps as in the differentiation w.r.t. a measure we did above.

Definition. Let ν be a signed measure and μ be a measure on $\mathcal{A}$. ν is called **absolutely continuous** w.r.t. μ, denoted by $\nu << \mu$, if $\nu(E) = 0$ for all $E \in \mathcal{A}$ with $\mu(E) = 0$.

Example. Let μ be a measure on a measurable space $(X, \mathcal{A})$ and f be an integrable function. Then the signed measure ν defined by $\nu(E) = \int_E f d\mu$, for all $E \in \mathcal{A}$, is absolutely continuous w.r.t. μ.

Exercises. 1. $\nu << \mu$ if and only if $|\nu| << \mu$ if and only if $\nu^+ << \mu$ and $\nu^- << \mu$.

2. If $\lambda_1 \perp \mu$ and $\lambda_2 \perp \mu$, then $\lambda_1 + \lambda_2 \perp \mu$.

3. If $\rho_1 << \mu$ and $\rho_2 << \mu$, then $\rho_1 + \rho_2 << \mu$.

Theorem.9 Let ν be a finite signed measure and μ be a measure on $(X, \mathcal{A})$. Then $\nu << \mu$ if and only if for every $\epsilon > 0$ there exists $\delta > 0$ such that $|\nu(E)| < \epsilon$ whenever $\mu(E) < \delta$.

Proof. By the exercise above, it is enough to prove the statement for $\nu = |\nu|$ is positive. The direction ($\Leftarrow$) is obvious from the definitions. So, we'll prove the converse by contrapositive. For, if $\epsilon - \delta$ condition is not satisfied, then there exists an $\epsilon > 0$ such that for all $n \in \mathbb{N}$ $\exists E_n \in \mathcal{A}$ with $\mu(E_n) < \frac{1}{2^n}$ and $\nu(E_n) \geq \epsilon$. Letting $F_k = \cup_{n=k}^{\infty}$ and $F = \cap_{k=1}^{\infty}$, we have $\mu(F_k) < \sum_{n \geq k} 2^{-n} = 2^{1-k}$; hence, $\mu(F) = 0$. On the other hand, $\nu(F_k) \geq \epsilon$ for all $k \geq 1$; hence, $\nu(F) = \lim_k \nu(F_k) \geq \epsilon$. But this contradicts with the fact that $\nu << \mu$. ■

Remarks. 1. Compare this theorem with Theorem.12 in Section III.3.

2. Finiteness of ν is essential. For otherwise, if $X = (0, 1)$, $\mathcal{A} = \mathcal{F}$, $\mu = m$, and ν is defined by $\nu(A) = \int_A \frac{1}{x} dm$, then the assertion of Theorem.9 does not hold. Another counterexample is given by $X = \mathbb{N}$, $\mathcal{A} = \mathcal{P}(\mathbb{N})$, μ is defined by $\mu(E) = \sum_{n \in E} \frac{1}{2^n}$, $\forall E \subset \mathbb{N}$, and ν =counting measure.

<u>Notation.</u> If $\nu(E) = \int_E f d\mu$, then this relationship can also be expressed by $d\nu = f d\mu$.

Lemma. Let ν and μ be finite measures on $(X, \mathcal{A})$. Then, either $\nu \perp \mu$, or there exists $\epsilon > 0$ and $E \in \mathcal{A}$ such that $\mu(E) > 0$ and E is a positive set for the measure $\nu - \epsilon\mu$.

Proof. Exercise. [Hint: Use Hahn decomposition for $\nu - n^{-1}\mu$.]

Theorem.10 (Radon-Nikodým) Let $(X, \mathcal{A}, \mu)$ be a finite measure space and $\nu : \mathcal{A} \to \mathbb{R}$ be a finite measure with $\nu << \mu$. Then there exists a non-negative $\mathcal{A}$-measurable function, unique μ-a.e., such that

$$\nu(E) = \int_E f d\mu, \quad \forall E \in \mathcal{A}.$$

Proof. (Uniqueness) If there exists non-negative $\mathcal{A}$-measurable functions f_1 and f_2 such that $\nu(E) = \int_E f_i d\mu, \quad \forall E \in \mathcal{A}, \ i = 1, 2$, then $0 = \int_E (f_1 - f_2) d\mu \ \forall E \in \mathcal{A}$. Hence, $f_1 = f_2$ μ-a.e. on X.

(Existence) If $\nu(E) = 0 \ \forall E \in \mathcal{A}$, the assertion holds with $f \equiv 0$. Hence, assume that ν does not vanish on all of $\mathcal{A}$. Define

$$\mathcal{G} = \{g : X \to [0, \infty] : \int_E g \le \nu(E), \ \forall E \in \mathcal{A}\}.$$

Since $0 \in \mathcal{G}$, we see that $\mathcal{G} \neq \emptyset$. Also, if $h, g \in \mathcal{G}$, then by letting $A = \{x : h(x) > g(x)\}$, for any $E \in \mathcal{A}$, it follows that

$$\int_E (h \vee g) d\mu = \int_{E \cap A} h d\mu + \int_{E \setminus A} g d\mu \le \nu(E \cap A) + \nu(E \setminus A) = \nu(E)$$

implying that $h \vee g \in \mathcal{G}$. Now, define $\alpha = \sup_{g \in \mathcal{G}} \int g d\mu$. Since $\alpha \le \nu(X) < \infty$, we can pick $\{g_n\} \subset \mathcal{G}$ such that $\int g_n \to \alpha$. Let $f_n = \max\{g_1, \ \ldots, g_n\}$, then $f_n \in \mathcal{G}$ for all n and $f_n \uparrow f$ pointwise, for some non-negative $\mathcal{A}$-measurable function f. Since $\int f_n d\mu \ge \int g_n d\mu$, it follows that $\int f_n \to \alpha$; and hence, by MCT, $f \in \mathcal{G}$ with $\int f d\mu = \alpha$ (in particular, $f < \infty$ μ-a.e.).

Next, define a new measure η by $\eta(E) = \nu(E) - \int_E f d\mu$ for all $E \in \mathcal{A}$. The, by the Lemma above, we have two cases:

$$(i) \ \eta \perp \mu, \quad \text{or} \quad (ii) \ \exists \epsilon > 0 \ \& \ A \in \mathcal{A} \text{ such that } (\eta - \epsilon\mu)(A) \ge 0 \ \& \ \mu(A) > 0.$$

In the first case, since $\eta << \mu$ by construction and $\nu << \mu$, it follows that $\eta \equiv 0$. Hence, $\nu(E) = \int_E f d\mu$ for all $E \in \mathcal{A}$, proving the assertion.

In the second case, we have $\nu(A) - \int_A f d\mu - \epsilon\mu(A) \ge 0$. Thus, $\nu(A) \ge \int_A f d\mu + \epsilon\mu(A) = \int_A (f + \hat{f}) d\mu$, where $\hat{f} = \epsilon\chi_A$. Since $\mu(A) > 0$, we have $\hat{f} \neq 0$ and $\int_A \hat{f} d\mu > 0$. Thus, $\int_X (f + \hat{f}) d\mu > \int_X f d\mu = \alpha$, contradiction. Hence (ii) is not possible, proving the theorem. ∎

Corollary. a) Theorem.10 is valid if both μ and ν are σ-finite.

b) Theorem.10 is valid if and ν is a σ-finite signed measure and μ is a σ-finite measure.

Proof. a) Let $X = \cup_i X_i$ and define $\mu_i(E) = \mu(X_i \cap E, \ E \in \mathcal{A}$, and $\nu_i(E) = \nu(X_i \cap E, \ E \in \mathcal{A}$. Apply Radon-Nikodým Theorem to $\{\mu_i, \ \nu_i\}$ to obtain f_i. Let $f = \sum_i f_i$.

b) Let $\nu = \nu^+ - \nu^-$ and apply (a) to $\{\mu, \ \nu^+\}$ and $\{\mu, \ \nu^-\}$ to obtain f^+ and f^-, respectively. Let $f = f^+ - f^-$. ∎

Notation. The function f in Radon-Nikodým Theorem and its Corollary is denoted by $\frac{d\nu}{d\mu}$ and is called as the **Radon-Nikodým derivative of** ν with respect to μ.

Theorem.11 (Lebesgue Decomposition of a measure) Let ν be a σ-finite signed measure and μ be a σ-finite measure on a measurable space $(X, \mathcal{A})$. Then there exists μ-a.e. unique σ-finite signed measures ν_0 and ν_1 on $\mathcal{A}$ such that $\nu_0 \perp \mu, \ \nu_1 << \mu$ and $\nu = \nu_0 + \nu_1$.

Proof. WLOG we can assume that both ν and μ are finite. First, let ν be a measure, and let $\lambda = \mu + \nu$, then $\mu << \lambda$. Hence, by Radon-Nikodým Theorem, there exists a λ-a.e. unique

$\mathcal{A}$-measurable function f such that $\mu(E) = \int_E f d\lambda$ for all $E \in \mathcal{A}$. Observe that f is also μ-a.e. and ν-a.e. $\mathcal{A}$-measurable function. Thus,

$$\mu(E) = \int_E f d\mu + \int_E f d\nu, \ \forall\ E \in \mathcal{A}. \ (*)$$

Let $X_1 = \{x : f(x) > 0\}$, $X_0 = \{x : f(x) = 0\}$, and define $\nu_0 = \nu|_{X_0}$ and $\nu_1 = \nu|_{X_1}$. Then we have:

(i) ν_0 and ν_1 are measures on $\mathcal{A}$.
(ii) $\nu = \nu_0 + \nu_1$.

Also, by construction, $X = X_0 \cup X_1$, $X_0 \cap X_1 = \emptyset$, and $\nu_0(X_1) = \nu_1(X_0) = 0$. Since by (*) $\mu(X_0) = \int_{X_0} f d\mu + \int_{X_0} f d\nu$, it follows that $\nu_0 \perp \mu$.

If $A \in \mathcal{A}$ such that $\mu(A) = 0$, then $\int_A f d\mu = 0$. So, by (*), we have

$$0 = \int_A f d\mu = \int_A f d\mu + \int_A f d\nu = \int_A f d\nu = \int_{A \cap X_0} f d\nu + \int_{A \cap X_1} f d\nu.$$

Since $f = 0$ on $A \cap X_0$ and $f > 0$ on $A \cap X_1$, we have $\int_{A \cap X_1} f d\nu = 0$, so $\nu(A \cap X_1) = 0$. Thus $\nu_1(A) = 0$, implying that $\nu_1 << \mu$.

If ν is a signed measure, then $\nu = \nu^+ - \nu^-$ where ν^+ and ν^- are measures. Then apply the above to ν^+ and ν^-, respectively, to obtain the decomposition $\nu^{\pm} = \nu_0^{\pm} + \nu_1^{\pm}$ with $\nu_0^{\pm} \perp \mu$, $\nu_1^{\pm} << \mu$. Then, $\nu = \nu_0 + \nu_1$, where $\nu_0 = \nu_0^+ + \nu_0^-$ and $\nu_1 = \nu_1^+ + \nu_1^-$. ∎

Remark. It is essential in Theorem.11 that ν and μ are σ-finite. [Exercise: Provide counterexamples when σ-finiteness in Theorem.11 is dropped.]

Exercises

1. Let μ be a probability measure on $(\mathbb{R}, \mathcal{B})$. Define a measure $\nu : \mathcal{B} \to \mathbb{R}$ by $\nu(E) = 1$ if $0 \in E$ and $\nu(E) = 0$ otherwise (ν is called the *point mass at* 0). Find the Jordan decomposition of the signed measure $\lambda = \mu - \nu$.

2. Let $(X, \mathcal{A}, \mu)$ be a measure space and ν_1 and ν_1 be signed measures on $\mathcal{A}$. Prove the following.

a) If $\nu_1 \perp \mu$ and $\nu_2 \perp \mu$, then $\nu_1 + \nu_2 \perp \mu$.
b) If $\nu_1 << \mu$ and $\nu_2 << \mu$, then $\nu_1 + \nu_2 << \mu$.
c) If $\nu_1 << \mu$, then $|\nu_1| << \mu$; and conversely.
d) If $\nu_1 << \mu$ and $\nu_2 \perp \mu$, then $\nu_1 \perp \nu_2$.
e) If $\nu_1 << \mu$ and $\nu_1 \perp \mu$, then $\nu_1 \equiv 0$.

3. Let $(X, \mathcal{A}, \mu)$ be a measure space, $\{E_k\}_{k=1}^n \subset \mathcal{A}$, and $\{c_k\}_{k=1}^n$ be a collection of real numbers. For $E \in \mathcal{A}$, define $\nu(E) = \sum_{k=1}^n c_k \cdot \mu(E \cap E_k)$. Show that $\nu << \mu$ and find $\frac{d\nu}{d\mu}$.

4. Let $\{\mu_n\}$ be a sequence of measures on a measurable space $(X, \mathcal{A})$ such that there is a constant C with $\mu_n(X) \leq C$ for all $n \geq 1$. Define $\nu : \mathcal{A} \to \mathbb{R}$ by

$$\nu(E) = \sum_{n=1}^{\infty} \frac{\mu_n(E)}{2^n}, \ E \in \mathcal{A}.$$

Show that ν is a measure on $\mathcal{A}$ and that $\mu_n << \nu$ for each $n \geq 1$.

5. Consider the measure space $([0,1], \mathcal{F}|_{[0,1]}, m)$ and let ν be the counting measure on $\mathcal{F}|_{[0,1]}$. Show that

a) $m << \nu$, and
b) there is no function $f : [0,1] \to \mathbb{R}$ for which $m(E) = \int_E f d\nu$ for all $E \in \mathcal{F}|_{[0,1]}$.

6. Let $\{a_n\}_n$ be a fixed sequence of real numbers and $\{p_n\}_n$ be a sequence of positive real numbers. Define a set function $\nu : \mathcal{F} \to \mathbb{R}$ by

$$\nu(E) = \sum_{a_n \in E} p_n, \quad E \in \mathcal{F}.$$

a) Show that ν is a σ-finite measure on $\mathcal{F}$.
b) Find the Lebesgue decomposition of ν with respect to the Lebesgue measure on $\mathcal{F}$.

7. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2 - 6x + 5$, and define a set function $\nu : \mathcal{B} \to \mathbb{R}$ by

$$\nu(E) = \int_E f(x) dm, \quad E \in \mathcal{B}.$$

a) Show that ν is a σ-finite measure on $\mathcal{B}$.
b) Find the Hahn decomposition of $\mathbb{R}$ with respect to ν.
c) Find the Jordan decomposition of ν.
d) Find the Lebesgue decomposition of ν with respect to the Lebesgue measure on $\mathcal{F}$.

8. Give an example in which the assertion of the Radon-Nikodỳm theorem fails (that is, $\lambda << \mu$ but there is no measurable function f such that $\lambda(A) = \int_A f d\mu$ for all measurable set A).

9. Let $(X, \mathcal{A}, \mu)$ be a measure space and g be a non-negative measurable function on X. Let a measure $\lambda : \mathcal{A} \to \mathbb{R}$ be defined as

$$\lambda(A) = \int_A g d\mu, \quad \text{for all } A \in \mathcal{A}.$$

Prove that if $f : X \to \mathbb{R}$ is a measurable function then

$$\int_X f d\lambda = \int_X f g d\mu,$$

in the sense that if one integral exists, so does the other, and the two integrals are equal.

10. Consider the measure space $([0,1], \mathcal{F}|_{[0,1]}, m|_{[0,1]})$. Define a set function λ on all subintervals $[a,b] \subset [0,1]$ by

$$\lambda([a,b]) = \phi(b) - \phi(a), \text{ where } \phi \text{ is the Cantor-Lebesgue function on } [0,1].$$

If μ is the measure obtained by restricting the outer measure λ^* to λ^*-measurable subsets of $[0,1]$, prove that $\mu \perp m|_{[0,1]}$.