FUNDAMENTALS OF REAL ANALYSIS
by

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V. PRODUCT MEASURE SPACES

V.1. Product measures

Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be two measure spaces. In this section we will construct a product measure $\mu \times \nu$ on $X \times Y$ that coincides with μ and ν when "restricted" to X and Y, respectively.

Definition. If $A \subset X$ and $B \subset Y$, the set $A \times B \subset X \times Y$ is called a **rectangle**; if $A \in \mathcal{A}$ and $B \in \mathcal{B}$, then $A \times B$ is called a **measurable rectangle**.

Fact.1 Let $\mathcal{R} = \{A \times B : A \in \mathcal{A},\ B \in \mathcal{B}\}$, the collection of of all measurable rectangles. Then:

a) $\mathcal{R}$ is a covering class for subsets of $X \times Y$.
b) If $U, V \in \mathcal{R}$, then $U \cap V \in \mathcal{R}$.
c) If $U \in \mathcal{R}$, then U^c is a finite disjoint union of sets in $\mathcal{R}$.
d) If $U, V \in \mathcal{R}$, then $U \cup V$ is a finite disjoint union of sets in $\mathcal{R}$.

Proof. Exercise.

Remark. (b)-(d) of Fact.1 says that the class $\mathcal{S}$ of all finite disjoint union of sets in $\mathcal{R}$ forms an algebra.

Definition. The σ-algebra generated by $\mathcal{S}$ is called the **product σ-algebra** (on $X \times Y$) and is denoted by $\mathcal{A} \times \mathcal{B}$.

Remark. By definition, $\mathcal{A} \times \mathcal{B}$ is the smallest σ-algebra containing $\mathcal{S}$.

On $\mathcal{A} \times \mathcal{B}$ one can define numerous measures; however, our aim is to define (construct) a measure which is "compatible" with μ and ν. For, we will follow the procedure that enabled us to construct the Lebesgue measure on $\mathbb{R}$ (with appropriate modifications, of course).

Given the measure spaces $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$, define a set function $\lambda : \mathcal{R} \to \mathbb{R}^{\#}$ by

$$\lambda(A \times B) = \mu(A)\nu(B), \quad A \in \mathcal{A},\ B \in \mathcal{B}.$$

Then $\lambda(\emptyset) = 0$, and is compatible with μ and ν in the sense that $\lambda(A \times Y) = \mu(A)$ and $\lambda(X \times B) = \nu(B)$.

Fact.2 Let $\mathcal{R}$ and λ be as above. Then,

a) λ is monotone on $\mathcal{R}$.
b) If $\{A_i \times B_i\}_{i=1}^{\infty}$ is a countable disjoint collection in $\mathcal{R}$ whose union is also in $\mathcal{R}$, say $A \times B = \cup_i A_i \times B_i$, then $\mu(A)\nu(B) = \sum_{i=1}^{\infty} \mu(A_i)\nu(B_i)$.

Proof. (a) Obvious.

(b) Fix $x \in A$. Then, for each $y \in B$, the point (x, y) belongs exactly one $A_i \times B_i$ by disjointness; the same also holds if x and y are interchanged. Thus, for any $x \in X$ and $y \in Y$,

$$\chi_A(x)\chi_B(y) = \chi_{A \times B}(x, y) = \sum_i \chi_{A_i \times B_i}(x, y) = \sum_i \chi_{A_i}(x)\chi_{B_i}(y).$$

Now, integrating w.r.t. x (i.e., μ) and using MCT, we obtain $\mu(A)\chi_B(y) = \sum_i \mu(A_i)\chi_{B_i}(y)$, and integrating this w.r.t. y (i.e., ν), it follows that $\mu(A)\nu(B) = \sum_{i=1}^{\infty} \mu(A_i)\nu(B_i)$. ∎

It follows from Fact.2 that the set function λ is also a countably additive set function on $\mathcal{S}$. Therefore, λ generates an outer measure $(\mu \times \nu)^*$ on subsets of $X \times Y$. Let $\mathcal{G}$ be the σ-algebra of $(\mu \times \nu)^*$-measurable subsets of of $X \times Y$ and let $\mu \times \nu = (\mu \times \nu)^*|_{\mathcal{G}}$. Then, $\mu \times \nu$ is a measure on $\mathcal{G}$. Notice that, since $\mathcal{A} \times \mathcal{B}$ is the smallest σ-algebra containing $\mathcal{S}$, we see that $\mathcal{A} \times \mathcal{B} \subset \mathcal{G}$; and hence, $\mu \times \nu$ is also a measure on $\mathcal{A} \times \mathcal{B}$.

Definition. The measure $\mu \times \nu$ is called the **product measure** of μ and ν, and $\mathcal{A} \times \mathcal{B}$ is called the **product σ-algebra** of $\mathcal{A}$ and $\mathcal{B}$. The triple $(X \times Y, \mathcal{A} \times \mathcal{B}, \mu \times \nu)$ is called the **product measure space** of $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$.

Remarks. 1. The triple $(X \times Y, \mathcal{G}, \mu \times \nu)$ is also a measure space; indeed, it is a complete measure space (see Section II. 5) (?). The difference between $\mathcal{G}$ and $\mathcal{A} \times \mathcal{B}$ is the sets of $\mu \times \nu$-measure zero.

2. If $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ are finite (σ-finite) measure spaces, then so is $(X \times Y, \mathcal{A} \times \mathcal{B}, \mu \times \nu)$.

3. If $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ are σ-finite, then $(\mu \times \nu)$ is the *unique* measure on $\mathcal{A} \times \mathcal{B}$ such that $(\mu \times \nu)(A \times B) = \mu(A)\nu(B)$ for all $A \times B \in \mathcal{R}$. If $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ are not σ-finite, then this assertion may not hold.

4. A typical element of $\mathcal{A} \times \mathcal{B}$ need not be a measurable rectangle; and union of elements of $\mathcal{G}$ need not be a measurable rectangle, either.

Terminology. If $X = Y = \mathbb{R}$, $\mathcal{A} = \mathcal{B} = \mathcal{F}$ and $\mu = \nu = m$, then $\mu \times \nu = m \times m$ is called the **two-dimensional Lebesgue measure**.

Exercise. Take two of your favorite measure spaces (they can be the same) and construct measures on a σ-algebra of subsets of their product (not necessarily compatible with the component measures).

V.2. Double and Iterated Integrals

Now, having product measure spaces defined, we will proceed to define integration on product measure spaces. For, we will continue following the process of defining Lebesgue measurable sets and construction of Lebesgue integral in Chapter.III.

In what follows $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ are (finite or σ-finite) measure spaces, then so is $\mathcal{G}$, $\mathcal{A} \times \mathcal{B}$, and $\mu \times \nu$ are as defined above.

Definition. A function $f : X \times Y \to \mathbb{R}^{\#}$ is called **$\mu \times \nu$-measurable** (or $\mathcal{A} \times \mathcal{B}$-measurable) iff $f^{-1}(O) \in \mathcal{A} \times \mathcal{B}$ for every open set $O \subset \mathbb{R}$. Similarly, $f : X \times Y \to \mathbb{R}^{\#}$ is called **$\mathcal{G}$-measurable** iff $f^{-1}(O) \in \mathcal{G}$ for every open set $O \subset \mathbb{R}$.

Now, as in Chapter.III, beginning with simple functions, we can define $\mu \times \nu$-integral of any $\mathcal{A} \times \mathcal{B}$-measurable (or $\mathcal{G}$-measurable) function. Define $f : X \times Y \to \mathbb{R}^{+}$ as $\mu \times \nu$-integrable if $\int_{X \times Y} f d(\mu \times \nu) < \infty$, and $f : X \times Y \to \mathbb{R}$ as $\mu \times \nu$-integrable if both f^{+} and f^{-} are $\mu \times \nu$-integrable.

Now, we will explore the ramifications of the property $(\mu \times \nu)(A \times B) = \mu(A)\nu(B)$, for all $A \times B \in \mathcal{R}$, of the measure $\mu \times \nu$ in the integral of any function $f : X \times Y \to \mathbb{R}^{\#}$.

Let $E \subset X \times Y$, $x \in X$ and $y \in Y$. Then we define

- $E_x = \{y : (x, y) \in E\}$ the x-section of E, and
- $E^y = \{x : (x, y) \in E\}$ the y-section of E.

Similarly, if $f : X \times Y \to \mathbb{R}^{\#}$ and $x \in X,\ y \in Y$, then we define

- $f_x(y) := f(x, y)$ (x is fixed) the x-section of f, and
- $f^y(x) := f(x, y)$ (y is fixed) the y-section of f.

Observe that, if $f : E \to \mathbb{R},\ E \subset X \times Y$, then $f_x(y)$ is a function on E_x and $f^y(x)$ is a function on E^y.

Remark. For any measurable rectangle $E = A \times B,\ x \in X$ and $y \in Y$,

$$E_x = (A \times B)_x = \begin{cases} B & \text{if } x \in A \\ \emptyset & \text{otherwise,} \end{cases} \qquad E^y = (A \times B)^y = \begin{cases} A & \text{if } y \in B \\ \emptyset & \text{otherwise.} \end{cases}$$

Exercise. For $\{E_n\}_n \subset \mathcal{A} \times \mathcal{B}$, show that $(\cup_n E_n)_x = \cup_n (E_n)_x$ and $(\cap_n E_n)_x = \cap_n (E_n)_x$. The same also hold for the y-sections of E_n's. Furthermore, for $E \subset \mathcal{A} \times \mathcal{B}$, $(E^c)_x = (E_x)^c$, the same also holds for y-section of E.

Fact.3 For $E \in \mathcal{A} \times \mathcal{B}$, we have (i) $E_x \in \mathcal{B}\ \forall x \in X$, and (ii) $E^y \in \mathcal{A}\ \forall y \in Y$.

Proof. It is enough to prove (i). Let $\mathcal{D} = \{E \in \mathcal{A} \times \mathcal{B} :\ E_x \in \mathcal{B}\ \forall x \in X\}$. By the Remark above, $\mathcal{D}$ contains all measurable rectangles. If $\{E_n\} \subset \mathcal{D}$, then $(\cup_n E_n)_x = \cup_n (E_n)_x \in \mathcal{B}\ \forall x \in X$. Thus, $\cup_n E_n \in \mathcal{D}$. Also, if $E \in \mathcal{D}$, then $(E^c)_x = (E_x)^c \in \mathcal{B}\ \forall x \in X$; hence, $E^c \in \mathcal{D}$. Since both $X \times Y$ and $\emptyset$ are in $\mathcal{D}$, we obtain that $\mathcal{D}$ is a σ-algebra. It follows that $\mathcal{A} \times \mathcal{B} \subset \mathcal{D}$, which implies (i). ∎

Remark. The same method of proof of Fact.3 shows that for any $E \in \mathcal{G},\ E_x \in \mathcal{B},\ \forall x \in X,$ and $E^y \in \mathcal{A},\ \forall y \in Y$.

Corollary. If $f : X \times Y \to \mathbb{R}$ is $\mathcal{A} \times \mathcal{B}$ -measurable, then f_x is $\mathcal{B}$-measurable for all $x \in X$ and f^y is $\mathcal{A}$-measurable for all $y \in Y$.

Proof. By Fact.3, we have $(f_x)^{-1}(O) = (f^{-1}(O))_x$ and $(f^y)^{-1}(O) = (f^{-1}(O))^y$ for any $O \subset \mathbb{R}$ open. ∎

Recall that $\mathcal{R}$ is the collection of all measurable rectangles in $X \times Y$, $\mathcal{S}$ is the algebra generated by $\mathcal{R}$ and $\mathcal{A} \times \mathcal{B}$ is the σ-algebra generated by $\mathcal{S}$ (hence, by $\mathcal{R}$).

Notation. By $\mathcal{R}_\sigma$ we will denote the collection of countable unions of sets in $\mathcal{R}$: and by $\mathcal{R}_{\sigma\delta}$ we will denote the collection of countable intersections of sets in $\mathcal{R}_\sigma$.

Fact.4 Let $E \in \mathcal{A} \times \mathcal{B}$ ($\mathcal{G}$). Then there exists $F \in \mathcal{R}_{\sigma\delta}$ such that $E \subset F$ and $(\mu \times \nu)(E) = (\mu \times \nu)(F)$.

Proof. Exercise. [Hint: See the proof of Theorem.2 in Chapter.II.]

Fact.5 Let $E \in \mathcal{A} \times \mathcal{B}$ ($\mathcal{G}$) such that $(\mu \times \nu)(E) < \infty$. Then

(i) the function $g(x) = \nu(E_x)$ is μ-measurable and $\int g d\mu = \int \nu(E_x) d\mu = (\mu \times \nu)(E)$, and
(ii) the function $h(y) = \nu(E^y)$ is ν-measurable and $\int h d\nu = \int \mu(E^y) d\nu = (\mu \times \nu)(E)$.

Proof. We'll prove (i) only, (ii) is left to the reader. By Fact.3 (i), $E_x \in \mathcal{B}$; hence, it is ν-measurable and $g(x) = \nu(E_x)$ is well-defined. If $E = A \times B,\ A \in \mathcal{A},\ B \in \mathcal{B}$, then the assertion is trivial.

Now, assume $E \in \mathcal{R}_\sigma$ such that $E = \cup_n E_n,\ E_n \in \mathcal{R}$, is a disjoint union. So $E + \cup_n (A_n \times B_n)$. Fix $x \in X$, then $E_x = \cup_n (A_n \times B_n)_x = \cup B_n$, and by countable additivity of $]nu$, we have

$$\nu(E_x) = \sum_n \nu(B_n) \sum_n \nu((A_n \times B_n)_x).$$

Let $g_n(x) = \nu((A_n \times B_n)_x)$ and $g(x) = \sum_n g_n(x)$. So, $g(x) = \nu(E_x)$. Since each g_n is μ-measurable, g is μ-measurable. Also, $g_n(x) \geq 0$ for all $n \geq 1$; hence, if $h_k(x) = \sum_{i=1}^k g_i(x)$, then $h_n \uparrow g$. Therefore, by MCT,

$$\int g d\mu = \lim_k \sum_1^k \int g_k d\mu = \lim_k \sum_1^k (\mu \times \nu)(A_k \times B_k) = (\mu \times \nu)(E),$$

which proves the assertion for sets in $\mathcal{R}_\sigma$.

If $E \in \mathcal{R}_{\sigma\delta}$, then there is $\{E_n\} \subset \mathcal{R}_\sigma$ such that $E_n \downarrow$ and $E = \cap_n E_n$. By hypothesis, $(\mu \times \nu)(E_1) < \infty$. Let $g_n(x) = \nu((E_n)_x)$, then $\int g_1 d\mu = (\mu \times \nu)(E_1) < \infty$, which implies that $g_1 < \infty$ μ-a.e. $x \in X$. Hence, for each $x \in X$ with $g_1(x) < \infty$, the family $\{E_n)_x\}$ is decreasing, each has finite ν-measure, and $E_x = \cap_n (E_n)_x$. Thus, by continuity of the measure ν,

$$g(x) = \nu(E_x) = \lim_n \nu((E_n)_x) = \lim_n g_n(x),$$

and hence, $g_n \to g$ μ-a.e. Since each g_n is μ-measurable, so is g. From $0 \leq g_n \leq g_1$ for all $n \geq 1$, applying DCT, and from the fact that $\int g_n d\mu = (\mu \times \nu)(E_n)$, we see that

$$\int g d\mu = \lim_n \int g_n d\mu = \lim[(\mu \times \nu)(E_n)] = (\mu \times \nu)(E),$$

proving the assertion for sets in $\mathcal{R}_{\sigma\delta}$. The assertion for arbitrary sets in $\mathcal{A} \times \mathcal{B}$ $(\mathcal{G})$ follows from Fact.4. ∎

Remark. By Fact.5, if $E \in \mathcal{A} \times \mathcal{B}$, then the functions $x \to \nu(E_x)$ and $y \to \mu(E^y)$ are measurable and

$$\int \nu(E_x) d\mu = (\mu \times \nu)(E) = \int \mu(E^y) d\nu.$$

Corollary. Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be σ-finite measure spaces. If if $E \in \mathcal{A} \times \mathcal{B}$, then the functions $x \to \nu(E_x)$ and $y \to \mu(E^y)$ are μ-measurable and ν-measurable, respectively, and

$$\int \nu(E_x) d\mu = (\mu \times \nu)(E) = \int \mu(E^y) d\nu.$$

Proof. Case $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ are finite measure spaces follow from Fact.5. For the general case, let $X \times Y = \cup_i (X_i \times Y_i)$, where $\{X_i \times Y_i\}_i$ is an increasing sequence of rectangles of finite measure. Then apply Fact.5 to $E \cap (X_i \times Y_i)$. ∎

For a $\mu \times \nu$-measurable function $f : X \times Y \to \mathbb{R}$, the integral of f w.r.t. $\mu \times \nu$ is defined as usual, i.e., $\int_{X \times Y} f(x, y) d(\mu \times \nu)$. We call this integral as the **double integral** of f. Since the the functions $f_x(y)$ and $f^y(x)$ are ν-measurable and μ-measurable functions, respectively, we can also consider the integrals $g(x) = \int_Y f_x(y) d\nu$ and $h(y) = \int_X f^y(x) d\mu$, if they are defined. In turn, if g and h are μ- and ν-measurable, respectively, then we can think about defining their integrals w.r.t. μ and ν, respectively, as

$$\int_X g(x) = \int_X [\int_Y f_x(y)d\nu]d\mu := \int_X \int_Y f(x,y)d\nu d\mu$$
$$\int_Y h(y) = \int_Y [\int_X f^y(x)d\mu]d\nu := \int_Y \int_X f(x,y)d\mu d\nu.$$

These are called the **iterated integrals** of f.

Now, observe that, if $\phi = \chi_E$ is the characteristic function of a set $E \in \mathcal{A} \times \mathcal{B}$ with finite measure, then by Fact.5, (i) for μ-a.e. $x \in X$, the function $\phi_x(y) = \phi(x,.)$ is ν-integrable on Y and the function $\phi^y(x) = \phi(.,y)$ is μ-integrable on X, respectively, and

$$\text{(ii)} \quad \int_{X\times Y} \phi d(\mu\times\nu) = \int_X (\int_Y \phi_x(y)d\nu)d\mu = \int_Y (\int_X \phi^y(x)d\mu)d\nu.$$

Consequently, the same also follows for simple functions $\phi : X \times Y \to \mathbb{R}$ vanishing outside sets of finite $\mu \times \nu$-measure.

Theorem.1 (Fubini's Theorem) Let Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be σ-finite measure spaces and let $f : X \times Y \to \mathbb{R}$ be a $\mu \times \nu$-integrable function. Then

a) for μ-a.e. $x \in X$, $f_x(y) := f(x,.)$ is ν-integrable,
b) for ν-a.e. $y \in Y$, $f^y(x) := f(.,y)$ is μ-integrable, and
c) $\int_{X\times Y} fd(\mu\times\nu) = \int_X(\int_Y f_x(y)d\nu)d\mu = \int_Y(\int_X f^y(x)d\mu)d\nu$.

Proof. WLOG we can assume that $f \geq 0$. Then there exists a monotone increasing sequence $\{\phi_n\}$ of simple functions such that $\phi_n \to f$ $\mu\times\nu$-a.e., where each ϕ_n vanish outside a set of finite $\mu \times \nu$-measure. Then, by the observation above, for each $n \geq 1$,

$$\int_{X\times Y} \phi_n d(\mu\times\nu) = \int_X [\int_Y (\phi_n)_x(y)d\nu]d\mu = \int_Y [\int_X (\phi_n)^y(x)d\mu]d\nu.$$

Now, by MCT,

$$(*) \quad \int_{X\times Y} fd(\mu\times\nu) = \lim_n \int_{X\times Y} \phi_n d(\mu\times\nu) = \lim_n \int_X [\int_Y (\phi_n)_x(y)d\nu]d\mu$$

and

$$(**) \quad \int_{X\times Y} fd(\mu\times\nu) = \lim_n \int_{X\times Y} \phi_n d(\mu\times\nu) = \lim_n \int_Y [\int_X (\phi_n)^y(x)d\mu]d\nu.$$

Fix $x \in X$. Then $\{(\phi_n)_x\}_n$ is a monotone increasing sequence of simple ν-measurable functions such that $(\phi_n)_x \to f_x$ ν-a.e. Hence, f_x is ν-measurable, and by MCT, we have $\int_Y f_x d\nu = \lim_n \int_Y (\phi_n)_x d\nu$. Next, let $h(x) = \int_Y f_x d\nu$ and $h_n(x) = \int_Y (\phi_n)_x d\nu$. Then, by the observation above, each h_n is μ-integrable. Since $h_n \uparrow h$ μ-a.e., by MCT, we obtain that

$$\int_X hd\mu = \lim_n \int_X h_n d\mu = \lim_n \int_X [\int_Y \phi_n d\nu]d\mu.$$

Since $\int_X hd\mu = \int_X[\int_Y f_x d\nu]d\mu$, this combined with (*) gives that

$$\int_{X\times Y} fd(\mu\times\nu) = \int_X [\int_Y f_x d\nu]d\mu.$$

Similarly, next by fixing $y \in Y$ and repeating the above for $\{(\phi_n)^y x\}_n$, we also obtain that $\int_{X\times Y} fd(\mu\times\nu) = \int_Y[\int_X f^y d\mu]d\nu$, proving the theorem. ∎

Remark. Fubini's Theorem requires f to be $\mu \times \nu$-integrable, which is not easy to show. Also, recall that, the usual integration process goes through defining integrals of measurable functions, and then define integrable functions. Tonelli's Theorem closes this gap.

Theorem.2 (Tonelli's Theorem) Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be σ-finite measure spaces and let $f : X \times Y \to \mathbb{R}$ be a non-negative $\mathcal{A} \times \mathcal{B}$-measurable function. Then

a) for μ-a.e. $x \in X$, $f_x(y) := f(x, .)$ is ν-measurable and the function $\int_Y f_x d\nu$ is μ-measurable,
b) for ν-a.e. $y \in Y$, $f^y(x) := f(., y)$ is μ-measurable and the function $\int_X f^y d\mu$ is ν-measurable, and
c) $\int_{X\times Y} f d(\mu \times \nu) = \int_X (\int_Y f_x(y) d\nu) d\mu = \int_Y (\int_X f^y(x) d\mu) d\nu$.

Proof. Since X and Y are σ-finite, so is $X \times Y$. Now, pick a sequence $\{f_n\}_n$ of simple functions that vanish outside a set of finite $\mu \times \nu$-measure such that $f_n \uparrow f$ pointwise. Hence, each f_n is $\mu \times \nu$-integrable. If $g_n(x) = \int_Y (f_n)_x d\nu$, then $g_n \uparrow f_x$ ν-a.e. Thus,

$$\int_X [\int_Y (f_n)_x d\nu] d\mu = \int_{X\times Y} f d(\mu \times \nu).$$

Now, the rest of the proof is the same as that of Fubini. ∎

Exercise. Let $X = Y = \mathbb{N}$, $\mathcal{A} = \mathcal{B} = \mathcal{P}(\mathbb{N})$, $\mu = \nu = c$, where c is the counting measure. Restate Fubini's and Tonelli's Theorems in this setting.

Remarks. 1. The integrals in Tonelli's Theorem can be ∞; however, if any one of the integrals in (c) is finite, so are the other two.

2. Typical application goes as follows: first, one applies Tonelli's Theorem and show that $\int_{X\times Y} f d(\mu \times \nu) < \infty$, and then uses Fubini to calculate the iterated integrals.

3. σ-finiteness in both Fubini and Tonelli's Theorems is essential as the following example shows.

Example. Consider $([0, 1], \mathcal{F}, m)$ and $([0, 1], \mathcal{P}([0, 1]), c)$, where c is the counting measure. Observe that $([0, 1], \mathcal{P}([0, 1]), c)$ is not a σ-finite measure space. Let $E = \{(x, y) : x = y\}$. Then

$$\int_{[0,1]} [\int_{[0,1]} \chi_E(x, y) dc] dm = 1 \neq 0 = \int_{[0,1]} [\int_{[0,1]} \chi_E(x, y) dm] dc.$$

(Also $\int_{[0,1]\times[0,1]} \chi_E(x, y) d(m \times c) = 0$.)

4. As the example below shows, non-negativity in Tonelli's Theorem is also essential.

Example. Let $X = Y = \mathbb{N}$, $\mathcal{A} = \mathcal{B} = \mathcal{P}(\mathbb{N})$, $\mu = \nu = c$, where c is the counting measure. Notice that, on $(\mathbb{N}, \mathcal{P}(\mathbb{N}, c)$, if $f : \mathbb{N} \to \mathbb{R}$, then f is measurable, and f is integrable if and only if $\sum_n |f(n)| < \infty$ (and $\int_{\mathbb{N}} f dc = \sum_n f(n)$). Now, consider a function $g : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ defined by

$$\begin{cases} g(n, n) = 1 & \forall n \geq 1 \\ g(n + 1, n) = -1 & \forall n \geq 1 \\ g(m, n) = 0 & \text{for } m \neq n, \; m \neq n + 1. \end{cases}$$

Hence, g is bounded and $c \times c$-measurable. So

$$\int_{\mathbb{N}} [\int_{\mathbb{N}} g(m, n) d\mu] d\nu = (1 - 1) + (1 - 1) + \cdots = 0,$$

$$\int_{\mathbb{N}} [\int_{\mathbb{N}} g(m, n) d\nu] d\mu = 1 + (-1 + 1) + (-1 + 1) + \cdots = 1.$$

This does not contradict with Tonelli's Theorem, nor Fubini's Theorem, since g is not non-negative. Also observe that

$$\int_{\mathbb{N}\times\mathbb{N}} |g| d(\mu \times \nu) = \int_{\mathbb{N}\times\mathbb{N}} g^+ d(\mu \times \nu) + \int_{\mathbb{N}\times\mathbb{N}} g^- d(\mu \times \nu),$$

but both the integrals on the RHS are infinite.

Exercises. 1. Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be σ-finite measure spaces and let $f : X \to \mathbb{R}$, $g : Y \to \mathbb{R}$ be $\mathcal{A}$- and $\mathcal{B}$-measurable functions, respectively. Then the function$h(x, y) = f(x)g(y)$ is an integrable function on $X \times Y$ and

$$\int_{X\times Y} h d(\mu \times \nu) = (\int_X f d\mu)(\int_Y g d\nu).$$

2. Let $X = Y = \mathbb{R}$, $\mathcal{A} = \mathcal{B} = \mathcal{F}$, and $\mu = \nu = m$. Let $f : \mathcal{R}^2 \to \mathcal{R}$ be defined by

$$f(x) = \begin{cases} \frac{x^2-y^2}{(x^2+y^2)^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Show that f is $m \times m$-measurable and

$$\int_{[0,1]} (\int_{[0,1]} f \; dm(x)) dm(y) \neq \int_{[0,1]} (\int_{[0,1]} f \; dm(y)) dm(x).$$

Remark. In general the product space $(X \times Y, \mathcal{A} \times \mathcal{B}, \mu \times \nu)$ is not complete, even if both $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ are complete. For instance, let $X = Y = \mathbb{R}$, $\mathcal{A} = \mathcal{B} = \mathcal{F}$, $\mu = \nu = m$. If $B \in \mathcal{P}(\mathbb{R}) \setminus \mathcal{F}$ and $A \in \mathcal{F}$, then $A \times B \notin \mathcal{A} \times \mathcal{B}$ by Fact.3. Since $A \times B \subset A \times \mathbb{R}$, we have $(\mu \times \nu)(A \times B) = (m \times m)(A \times B)$ is not defined, whereas, $\mu \times \nu)(A \times Y) = (m \times m)(A \times \mathbb{R}) = 0$.

However, we have the following:

Proposition. Let $(X, \mathcal{A}, \mu)$ be a (σ-finite) measure space, $(Y, \mathcal{B}, \nu)$ be a complete a (σ-finite) measure space, and $E \in \mathcal{G}$ be such that $(\mu \times \nu)(E) = 0$. Then for μ-a.e. $x \in X$, the set E_x is ν-measurable and $\nu(E_x) = 0$. [Hence $(\mu \times \nu)(E) = \int_X \nu(E_x) d\mu$.]

Proof. Since $(\mu \times \nu)(E) < \infty$, there exists $A \in \mathcal{R}_{\sigma\delta}$ such that $E \subset A$ and $(\mu \times \nu)(A) = (\mu \times \nu)(E) = 0$. By Fact.4, for all $x \in X$, A_x is ν-measurable and $0 = (\mu \times \nu)(A) = \int_X \nu(A_x) d\mu$. Thus $\nu(A_x) = 0$ for μ-a.e. $x \in X$. Since $E_x \subset A_x$ for all $x \in X$, and since $(Y, \mathcal{B}, \nu)$ is complete, for μ-a.e. $x \in X$, the set E_x is ν-measurable and $\nu(E_x) = 0$. ∎

Question. Let $X = Y = \mathbb{R}$, $\mathcal{A} = \mathcal{B} = \mathcal{F}$, $\mu = \nu = m$, and $E = \{a\} \times [0, 1]$, for some $a \in \mathbb{R}$. Then $(\mu \times \nu)(E) = 0$, but $\nu(E_a) \neq 0$. Does this contradict Proposition above?

Although the product space $(X \times Y, \mathcal{A} \times \mathcal{B}, \mu \times \nu)$ is not complete in general, the product space $(X \times Y, \mathcal{G}, \mu \times \nu)$ is complete, by construction. Actually the σ-algebra $\mathcal{G}$ is the completion of $\mathcal{A} \times \mathcal{B}$. Despite this fact, the relationship between a $\mathcal{G}$-measurable function and its x-section f_x and y-section f^y is much more complicated; hence we will not deal with it here. Instead, we will simply state and prove the version of Fubini-Tonelli Theorem in this setting.

Theorem.3 (Fubini-Tonelli Theorem for complete spaces) Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be complete σ-finite measure spaces and let $f : X \times Y \to \mathbb{R}$ be a non-negative $\mathcal{G}$-measurable function. Then

a) for μ-a.e. $x \in X$, $f_x(y) := f(x, .)$ is ν-measurable and the function $\int_Y f_x d\nu$ is μ-measurable,

b) for ν-a.e. $y \in Y$, $f^y(x) := f(.,y)$ is μ-measurable and the function $\int_X f^y d\mu$ is ν-measurable, and

c) $\int_{X\times Y} fd(\mu\times\nu) = \int_X(\int_Y f_x(y)d\nu)d\mu = \int_Y(\int_X f^y(x)d\mu)d\nu$.

If $f : X \times Y \to \mathbb{R}$ is $\mu \times \mu$-integrable (not necessarily non-negative), then in the statements (a) and (b) above measurable is replaced by integrable and (c) holds with finite value.

Proof. Exercise. [Hint: Follow the steps of the proof of Theorem 2 and Theorem.2 while using Proposition.]

Remarks. 1. The construction of product measure can be repeated to obtain product measure space of n (indeed, infinitely many) measure spaces. The versions of Fubini and Tonelli's Theorems are also valid in their respective settings.

2. The special case, when $X = Y = \mathbb{R}$, $\mathcal{A} = \mathcal{B} = \mathcal{F}$, $\mu = \nu = m$ (or when $X_i = \mathbb{R}$, $\mathcal{A}_i = \mathcal{F}$, $\mu_i = m$) is called **two (n) dimensional Lebesgue measure space.** All the properties above, of course, are valid in this case; furthermore, like the Lebesgue measure, the n-dimensional Lebesgue measure, $n \geq 2$, is invariant under congruent maps (i.e., under translations, reflections, and rotations) on $\mathbb{R}^n$.

Exercise. Consider the function $f(x,y) = e^{-xy} - 2e^{-2xy}$. Prove that:

a) $\int_{[0,1]}\int_{[1,\infty)} f(x,y)dm(x)dm(y) < \infty$.

b) $\int_{[1,\infty)}\int_{[0,1]} f(x,y)dm(y)dm(x) < \infty$.

Observe that $\int_{[0,1]}\int_{[1,\infty]} f(x,y)dm(x)dm(y) \neq \int_{[0,1]}\int_{[1,\infty]} f(x,y)dm(y)dm(x)$. Does this contradict to the Fubini-Tonelli Theorem? Explain.