FUNDAMENTALS OF REAL ANALYSIS
by

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VI. SPECIAL TOPICS

VI.1. A Brief Survey of L_p-spaces

Let $(X, \mathcal{A}, \mu)$ be a measure space. Throughout this section we will say that two functions $f, g : X \to \mathbb{R}$ are *equivalent* if $f = g$ μ-a.e. For any measurable function $f : X \to \mathbb{R}$ the class of all measurable functions equivalent to f will be denoted by $[f]$. Consequently, $f \in [f]$, and $g, h \in [f] \iff \int_X |f - g| d\mu = 0$. We will write f instead of $[f]$ for notational convenience.

For any real number $1 \leq p < \infty$, we will denote by $L_p(X, \mathcal{A}, \mu) = L_p(X)$ the class of all measurable functions f such that $|f|^p$ is integrable. If

$$\|f\|_p = [\int |f|^p d\mu]^{1/p},$$

then we see that $L_p(X) = \{f : \|f\|_p < \infty\}$.

For any measurable function $f : X \to \mathbb{R}$, let

$$ess \sup_X |f(x)| := \inf\{M : \mu(\{x \in X : f(x) > M\}) = 0\}.$$

If $\|f\|_\infty = ess \sup_X |f(x)|$, then we define

$$L_\infty(X) = \{f : \|f\|_\infty < \infty\}.$$

Exercises. 1. Under the usual (pointwise) addition of functions and multiplication of functions by a scalar, the function classes $L_p(X)$ form a vector space for each $1 \leq p \leq \infty$. [Hint: For $1 < p < \infty$, observe that $|f + g|^p \leq 2^p(|f|^p + |g|^p)$.]

2. Show that $\|.\|_1$ and $\|.\|_\infty$ define norms on $L_1(X)$ and $L_\infty(X)$, respectively. Hence $(L_1(X), \|.\|_1)$ and $(L_\infty(X), \|.\|_\infty)$ are normed spaces.

It might be surmised that, for $1 < p < \infty$, the vector spaces $L_p(X)$ are normed spaces as well. This assertion is correct; however, one needs some extra work for that end.

Lemma. For any $a \geq 0,\ b \geq 0$ and $1 < p < \infty$, we have

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}, \quad \text{where } \frac{1}{p} + \frac{1}{q} = 1.$$

Proof. Consider the function $s(x) = \frac{x^p}{p} + \frac{x^{-q}}{q}$ on $[0, \infty)$. Then s is strictly decreasing on $(0, 1]$ and stricly increasing on $[1, \infty)$ with $s(1) = 1$. Hence $\frac{x^p}{p} + \frac{x^{-q}}{q} \geq 1$ for all $x > 0$. When $a > 0,\ b > 0$, taking $x = \frac{a^{1/q}}{b^{1/p}}$ and multiplying the resulting inequality by ab, we obtain the assertion in this case. Observe that s is continuous; hence the assertion for any $a \geq 0,\ b \geq 0$ follows. ∎

Theorem. (Hölder's Inequality) Let $p, q \in [0, \infty]$ such that $\frac{1}{p} + \frac{1}{q} = 1$ (with the convention that $p = \infty$ if $q = 1$ and $q = \infty$ if $p = 1$). If $f \in L_p(X)$ and $g \in L_q(X)$, then $fg \in L_1(X)$ and

$$\|fg\|_1 \leq \|f\|_p \ \|g\|_q.$$

Proof. If $f = 0$ or $g = 0$, then the inequality is trivially valid; hence assume that $\|f\|_p \neq 0$ and $\|g\|_q \neq 0$. The assertion is also valid for $p = \infty,\ q = 1$ or $q = \infty,\ p = 1$; hence we will assume that $1 < p, q < \infty$. In that case, letting $a = \frac{|f(x)|}{\|f\|_p}$ and $b = \frac{|g(x)|}{\|g\|_q}$ in the Lemma above, we obtain

$$\frac{|f(x)g(x)|}{\|f\|_p\|g\|_q} \leq \frac{1}{p}\frac{|f(x)|}{\|f\|_p^p} + \frac{1}{q}\frac{|g(x)|}{\|g\|_q^q}.$$

The assertion follows by integrating both sides of this ineqality. ∎

Theorem. (Minkowski's Inequality) Let $p \in [0, \infty]$. If $f, g \in L_p(X)$, then $f + g \in L_p(X)$ and $\|f + g\|_p \leq \|f\|_p + \|g\|_p$.

Proof. The assertion for $p = 1$ or $p = \infty$ is easy, so we assume $1 < p < \infty$. We already know that $|f + g| \in L_p(X)$. Now,

$$\begin{aligned}\|f+g\|_p^p &= \int |f+g|^p = \int |f+g|^{p-1}|f+g| \leq \int |f||f+g|^{p-1} + \int |g||f+g|^{p-1} \\ &\leq [\int |f|^p]^{1/p}[\int |f+g|^p]^{1/q} + [\int |g|^p]^{1/p}[\int |f+g|^p]^{1/q} \text{ by Hölder's Inequality} \\ &= [\|f\|_p + \|g\|_q]\|f+g\|_p^{p/q}.\end{aligned}$$

This implies the assertion. ∎

Corollary. For $p \in [0, \infty]$, the space $(L_p(X), \|.\|_p)$ is a normed space.

Recall that every norm $\|.\|$ defines a metric ρ on any vector space X by $\rho(u, v) = \|u - v\|$. Hence, each $L_p(X)$ is also a metric space with metric $\rho(f, g) = \|f - g\|_p$. This metric space possesses the following important features, which we will state without proof. For proofs one can refer to any one of the references.

Theorem. If $1 \leq p \leq \infty$, the normed space $L_p(X)$ is complete.

Let $S_c := \{f : X \to \mathbb{R} : f$ simple and vanishes outside a set of finite measure$\}$. It is easy to see that $S_c \subset L_p(X)$ and $(S_c, \|.\|_p)$ is a normed space for $1 \leq p \leq \infty$. Indeed, we have even more:

Proposition. $(S_c, \|.\|_p)$ is dense in $(L_p(X)\|.\|_p)$.

Remark. In general, S_c need not be countable; however, if $X \subset \mathbb{R}$ and $\mu = m$, then it contains a countable dense subset, namely

$$S_s := \{f : X \to \mathbb{R} : f \text{ is a step function vanishing outside a set of finite measure}\}.$$

Consequently, for any $X \subset \mathbb{R}$, the metric space $L_p(X)\|.\|_p),\ \ 1 \leq p < \infty$, is **separable.** Another way to see this fact is via Weierstrass Approximation Theorem when X=[a,b] for some $a, b \in \mathbb{R}$. Let $C([a, b]) := \{f : [a, b] \to \mathbb{R} : f$ is continuous$\}$, then it is easy to show that $C([a, b])$ is a dense subset of $L_p([a, b])$. The fact that the polynomials on $[a, b]$ with rational coefficients is a dense subset of $C([a, b])$ implies that $L_p([a, b])$ is separable. Note that we not included the case $p = \infty$ in above, this is due to the fact that $L_\infty(X)$ is not separable.

In general there is no set theoretical "order" relationships between various L_p-spaces; however, if $\mu(X) < \infty$, then an easy application of H"older's Inequality shows that such an order exists.

Theorem. Let $(X, \mathcal{A}, \mu)$ be a finite measure space, and $1 \leq p < q \leq \infty$, then $L_q(X) \subset L_p(X)$.

Exercise. Give an example that the inclusion in the Theorem above is indeed proper.

Exercise. Let $X = \mathbb{N}$, $\mathcal{A} = \mathcal{P}(\mathbb{N})$, $\mu = c$, where c is the counting measure. Notice that, on $(\mathbb{N}, \mathcal{P}(\mathbb{N}, c)$, if $f : \mathbb{N} \to \mathbb{R}$, then f is measurable, and f is integrable if and only if $\sum_n |f(n)| < \infty$ (and $\int_{\mathbb{N}} f dc = \sum_n f(n)$). Restate the conditions under which any $f : \mathbb{N} \to \mathbb{R}$ is in $L_p(\mathbb{N})$, as well as Hölder's and Minkowski's inequalities in this setting.

VI.2. Modes of Convergence.

Having defined convergence of sequence of functions in several modes, in this section we will investigate the relationships among these types of convergence, and in the meantime, introduce two more modes of convergence. First, let's recall the types of convergences of sequences of functions studied so far in this course. For convenience, we will state them for sequences $\{f_n\} \subset L_p(X)$ of $\mathbb{R}$-valued functions for some measure space $(X, \mathcal{A}, \mu)$ and $1 \le p \le \infty$.

$f_n \to f$ **uniformly** if $\forall \epsilon > 0, \exists N \in \mathbb{Z}^+$, s.t. $n \ge N \Rightarrow |f_n(x) - f(x)| < \epsilon, \forall x \in X$.
$f_n \to f$ **pointwise** if $\forall x \in X$ and $\forall \epsilon > 0, \exists N \in \mathbb{Z}^+$ s.t. $n \ge N \Rightarrow |f_n(x) - f(x)| < \epsilon$.
$f_n \to f$ **almost uniformly** if $\forall \epsilon > 0, \exists A \subset X,\ \mu(A) < \epsilon$ s.t. $f_n \to f$ uniformly on A^c.
$f_n \to f$ **almost everywhere** if $\exists A \subset X,\ \mu(A) = 0$ s.t. $f_n \to f$ pointwise on A^c.
$f_n \to f$ **in measure** if $\forall \epsilon > 0,\ \lim_n \mu(\{x \in X : |f_n(x) - f(x)| > \epsilon\}) = 0$.

Now, having studied properties of L_p-spaces, we would like to introduce two new types of convergence.

Definition. A sequence of $\mathbb{R}$-valued functions $\{f_n\} \subset L_p(X)$ is said to **converges in L_p-norm (or in L_p, or in p-norm) to** $f \in L_p(X)$ if $\forall \epsilon > 0, \exists N \in \mathbb{Z}^+$, s.t. $n \ge N \Rightarrow \|f_n(x) - f(x)\|_p < \epsilon$.

Definition. A sequence of $\mathbb{R}$-valued functions $\{f_n\} \subset L_p(X)$ is said to **converge weakly to** $f \in L_p(X)$ if $\lim_n \int f_n g d\mu = \int f g d\mu,\ \forall g \in L_q(X)$, where $\frac{1}{p} + \frac{1}{q} = 1$.

Notation. In the following we will adopt the following notations:

$f_n \to^u f$, $f_n \to^p f$, $f_n \to^{a.u} f$, $f_n \to^{a.e} f$, $f_n \to^m f$, $f_n \to^{n_p} f$, and $f_n \to^w f$, will denote convergence of $\{f_n\}$ to f uniformly, pointwise, almost uniformly, almost everywhere, in measure, in L_p-norm, and weakly, respectively; and $[u]$, $[p]$, $[a.u]$, $[a.e.]$, $[m]$, $[n_p]$, and $[w]$ will denote uniform, pointwise, almost uniform, almost everywhere, in measure, L_p-norm, and weak convergence, respectively.

Now, clearly have the following immediate (obvious or proven before) relationships:

$$[u] \Rightarrow [a.u], \quad [u] \Rightarrow [p], \quad [a.e.] \Rightarrow [m].$$

Fact.1 (i) $[n_p] \Rightarrow [m]$, and (iii) $[n_p] \Rightarrow [w]$.

Proof. (i) Exercise. [Hint: Use Chebychev's Inequality.]

(ii) By Hölder's Inequality, $|\int f_n g d\mu - \int f g d\mu| \le \|f_n - f\|_p\ \|g\|_q \to 0$ by hypothesis. ∎

The following fact states that immediate relationships when X has finite measure; first implication is a restatement of Egoroff's Theorem and the second one is known.

Fact.2 If $\mu(X) < \infty$, then: (i) $[a.e.] \Rightarrow [a.u.]$, and (ii) $[u] \Rightarrow [n_p]$.

The converses of all the implications mentioned above are not valid in general. However, under additional (mild) conditions some of the converses are valid.

Fact.3 (i) $[p] \Rightarrow [u]$ when $f_n \le f_{n+1}$ and X is compact.

(ii) $[m] \Rightarrow [a.e.]$ along a subsequence.

Proof. (i) This is Dini's Theorem, covered in Math 451.

(ii) Chapter II, Theorem.15. ∎

In general, there is no direct relationships between [a.e] and $[n_p]$; however, they are related under some conditions.

Fact.4 (i) If $f_n \to f$ a.e., then $\|f_n - f\|_p \to 0$ if and only if $\|f_n\|_p \to \|f\|_p$. (Hence, $[a.e.] \Rightarrow [n_p]$ if $\|f_n\|_p \to \|f\|_p$.)

(ii) $[n_p] \Rightarrow [a.e.]$ along a subsequence.

Proof. (i) Apply Generalized DCT to $\{f_n\}$ and $\{|f_n|\}$.

(ii) Immediate from the facts that $[n_p] \Rightarrow [m]$ and $[m] \Rightarrow [a.e.]$ along a subsequence. ∎

Remark. The DCT (BCT)and MCT can be viewed as statements providing conditions for the implication $[a.e.] \Rightarrow [n_p]$:

[MCT] If $f_n \geq 0$ and $f_n \uparrow f$ a.e., then $f_n \to^{n_p} f$.

[DCT] Let $\exists g \in L_p^+$ s.t. $|f_n| \leq g$ for all $n \geq 1$. If $f_n \to^{a.e.} f$, then $f_n \to^{n_p} f$.

Exercise. Show by examples that the following implications fail:

$$[w] \Rightarrow [a.u.], \quad [w] \Rightarrow [n_p], \quad [w] \Rightarrow [a.e.], \quad [n_p] \Rightarrow [a.e.], \quad [m] \Rightarrow [a.e.], \quad [p] \Rightarrow [u].$$

Exercise. Let $X = \mathbb{N},\ \mathcal{A} = \mathcal{P}(\mathbb{N}),\ \mu = c$, where c is the counting measure.

a) Restate $[u],\ [a.u.],\ [p],\ [a.e.],\ [m],\ [n_p]$, and $[w]$ in this setting.
b) Restate BCT, DCT, Fatou's Lemma and MCT for functions in $L_p(\mathbb{N})$ in this setting.

VI.3 Littlewood's Three Principles

Although the theory of Lebesgue measure (and measurable functions) is developed to address the drawbacks of the Reimann's theory of integration, it is rather non-trivial as well as dealing with large classes of sets and functions, some of which are quite non-intuitive. In order to clarify some of the delicate results on these sets/functions, J.E. Littlewood, a well-known mathematician of early 20th century, came up with three principles that describes typical measurable sets, measurable functions and an a.e. convergent sequence of functions.

These three principles, that capture much of the essence of the Lebesgue's theory, are as follows:

1. Any measurable set is nearly a union of intervals. [$E \in \mathcal{F} \Rightarrow$ for all $\epsilon > 0$, there exists a disjoint union $A = \bigcup_{k=1}^{n} I_k$ of intervals I_k s.t. $m(A \setminus E) < \epsilon$].
2. Any measurable function is a nearly continuous function.
3. Any almost everywhere convergent sequence $\{f_n\}$ of functions on a set $E \in \mathcal{F}$ with $m(E) < \infty$ is nearly uniformly convergent.

The first principle is nothing but rephrasing of Theorem.2 in Section II.2. Similarly, the second one is rephrasing of a Theorem of Lusin:

Theorem. (Lusin) If $E \in \mathcal{F}$, $f : E \to \mathbb{R}$, then, for all $\epsilon > 0$, there exists $g : \mathbb{R} \to \mathbb{R}$ continuous and there exists a closed set $F \subset E$ such that $m(E \setminus F) < \epsilon$, $f = g$ on F.

The last one is also a rephrasing of another theorem due to Egoroff:

Theorem. (Egoroff) Let $E \in \mathcal{F}$, $m(E) < \infty$, $f,\ f_n : E \to \mathbb{R}$ measurable, $f_n \to f$ almost everywhere. Then for all $\epsilon > 0$ there exists a closed $F \subset E$ such that $m(E \setminus F) < \epsilon$ and $f_n \to f$ uniformly.

Egoroff's Theorem is proved in in Section III.3 (Theorem.5). We will not prove Lusin's Theorem here; an interested reader is referred to Royden's *Real Analysis* textbook, pp:64-67, for its proof. It should be noted here that, the original statement of Egoroff's Theorem requires $f_n \to f$ pointwise. Also, one can extend Lusin's Theorem to functions on a measurable set with infinite measure.

VI.4. Hausdorff Measure and Hausdorff Dimension

Measuring the "size" of mathematical objects has been one of the main objectives in mathematics throughout its history. For simple geometric objects, such as line segments, regular polygons, polyhedra, etc. the (topological) "dimension" and "measure" are adequate tools for this purpose. Although these two concepts serve well for this and other purposes (as seen in the previous five chapters), however, there are many other objects that neither topological dimension nor measure do not provide any valuable information about their "size." A typical example of such an object is the *Sierpinsky gasket.* This geometric object is an example of a *fractal*, whose "size", whether in terms of dimension or "volume" does require a more refined concept of measure and of dimension. In this section we will have a brief discussion of the most general, as well as most important, kind of such measures/dimensions: the *Hausdorff measure and Hausdorff dimension.* For more information and for the proofs of some of the statements below, the reader can refer to **Lectures on Fractal Geometry and Dynamical Systems,** by Y. Pesin and V. Climenhaga, AMS Student Mathematical Library, Vol. 52, 2009.

Consider the Euclidean space, i.e., $\mathbb{R}^n$ with the usual metric $d(x, y) := \|x - y\|_2$. For any (bounded) set $U \subset \mathbb{R}^n$, let

$$|U| = \text{ diameter of } U := \sup_{x,y \in U} d(x, y).$$

Let $A \subset \mathbb{R}^n$. For any $\delta > 0$, let *δ-covering of A* be a countable open cover $\mathcal{U} = \{U_n\}_{n \geq 1}$ of A such that $|U_n| < \delta$ for all $n \geq 1$. Observe that if $\delta_1 > \delta_2 > 0$, then any δ_2-cover is also a δ_1-cover.

For any $s \geq 0$, define

$$H^s_\delta(A) := \inf\{\sum_{n=1}^{\infty} |U_n| : \ \{U_n\} \text{ is a } \delta - \text{cover}\}.$$

Then it follows from the observation above that, as $\delta \uparrow$, we have $H^s_\delta(A) \downarrow$. Thus, since $H^s_\delta(.)$ is a monotonic, the limit

$$H^s(A) = \lim_{\delta \to 0} H^s_\delta(A) \text{ exists.}$$

Hence, H^s is a well-defined set function on subsets of $\mathbb{R}^n$; indeed, it is an outer measure on (subsets of) $\mathbb{R}^n$. Furthermore, every Borel set in $\mathbb{R}^n$ is H^s-measurable (Exercise). Let $\mathcal{Z}$ be the σ-algebra of H^s-measurable subsets of $\mathbb{R}^n$.

Definition. The set function $H^s : \mathcal{Z} \to [0, \infty]$ is called the **s-dimesional Hausdorff measure.**

Remarks. 1. $\mathcal{B}(\mathbb{R}^n) \subset \mathcal{Z}$.

2. H^s depends on $s \in [0, \infty)$ and the underlying metric d.

Fact.1 a) For $s = 0$, H^s is the counting measure; for $s = 1$, H^s is the Lebesgue measure m.

b) For $s = n$, $n \in \mathbb{Z}^+$, $H^s = Cm^n$, where m^n is the n-dimensional Lebesgue measure and C is a constant. Hence, H^s generalizes the (n-dimensional) Lebesgue measure.

Fora given value of s, the function $H^s(A)$ can take values in $[0, \infty]$; hence, it can be 0, ∞ or $0 < H^s(A) < \infty$. Typically, the values 0 and ∞ are not of interest. When $0 < H^s(A) < \infty$; however, we have the following observations.

Fact.2 If $s \geq 0$ and

a) if $H^s(A) < \infty$, then $H^t(A) = 0$ for all $t > s$,
b) if $H^s(A) > 0$, then $H^t(A) = \infty$ for all $t < s$.

As a consequence of Fact.2 (a) and (b), as a function of s, $H^s(A)$ takes ∞ for some s and after a threshold value of s we have $H^s(A) = 0$. This critical threshold value s_0 completely determines the function $H^{(\cdot)}(A)$; more explicitly,

$$H^s(A) = \begin{cases} \infty & \text{if } 0 \leq s < s_0 \\ 0 & \text{if } s_0 < s. \end{cases}$$

This critical value s_0 at which the function $H^{(\cdot)}(A)$ passes from ∞ to 0 is called the **Hausdorff dimension of** A, and is denoted by $dim_H(A)$. Hence, formally speaking,

$$dim_H(A) = \sup\{s \in [0, \infty) : H^s(A) = \infty\} = \inf\{s \in [0, \infty) : H^s(A) = 0\}.$$

Observe that, for a polygon P (in $\mathbb{R}^2$) or polyhedra Q (in $\mathbb{R}^3$), we have $m(P) = m^2(Q) = \infty$, and $m^3(P) = m^4(Q) = 0$, which suggest that $1 < dim_H(P) < 3$, and $2 < dim_H(Q) < 4$. Indeed, it turns out that for such "regular" objects in $\mathbb{R}^n$, topological dimension and Hausdorff dimension do agree (and has an integer value). However, the real flavor (and the distinction) of Hausdorff dimension reveals itself when the geometric object is an "irregular" one, like a fractal.

Examples. 1. Let $\mathcal{C}$ be the standard Cantor set. Then $H^{s_0}(\mathcal{C}) = 1$ if $s_0 = \frac{\log 2}{\log 3}$; hence, it follows that $dim_H(\mathcal{C}) = \frac{\log 2}{\log 3}$.

2. Let $\mathcal{S}$ be the Sierpinski gasket. Then $H^{s_0}(\mathcal{S}) = 1$ if $s_0 = \frac{\log 3}{\log 2}$; hence, it follows that $dim_H(\mathcal{S}) = \frac{\log 3}{\log 2}$.

Notice that both the Cantor set and the Sierpinski gasket, which are some prime examples of fractals, have non-integer values as their Hausdorff dimensions. Furthermore, their topological dimensions are 0 and 1, respectively. Due to this nature of their dimensions, some authors define fractals as objects whose Hausdorff dimension strictly exceeds the topological dimension (which is always an integer).

If $A = \{a\}$ any singleton set in $\mathbb{R}^n$, then $H^s_\delta(A) = 0$ for all $\delta > 0$, $s > 0$; hence $H^s(A) = 0$ for all $s > 0$, and consequently, $dim_H(A) = 0$. In order to have an idea about the Hausdorff dimension of other objects, we need the following.

Proposition. The Hausdorff dimension has the following properties:

a) $dim_H(\emptyset) = 0$.
b) If $A \subset B$, then $dim_H(A) \leq dim_H(B)$.
c) If $\{A_k\}_k$ is a countable collection of subsets of $\mathbb{R}^n$, then $dim_H(\cup A_k) = \sup\{dim_H(A_k)\}$.

Now, the following is immediate from the property (c) above:

Corollary. If $A \subset \mathbb{R}^n$ is a countable set, then $dim_H(A) = 0$.

Although there is no direct relationship between the topological dimension and Hausdorff dimension of a set, we still have some connections. For notice that $dim_H(\mathcal{C}) < 1$ and that $\mathcal{C}$ is totally disconnected subset of $\mathbb{R}$. This fact is not peculiar to $\mathcal{C}$ only, since have the following interesting result:

Fact.3 If $A \subset \mathbb{R}$ such that $dim_H(A) < 1$, then A is totally disconnected.