Math 750
REAL ANALYSIS

Assignment #6; Due on Dec. 12, 2014

1. Let μ be a probability measure on $(\mathbb{R}, \mathcal{B})$. Define a measure $\nu : \mathcal{B} \to \mathbb{R}$ by $\nu(E) = 1$ if $0 \in E$ and $\nu(E) = 0$ otherwise (ν is called the *point mass at* 0). Find the Jordan decomposition of the signed measure $\lambda = \mu - \nu$.

2. Let $(X, \mathcal{A}, \mu)$ be a measure space and ν_1 and ν_1 be signed measures on $\mathcal{A}$. Prove the following.

a) If $\nu_1 \perp \mu$ and $\nu_2 \perp \mu$, then $\nu_1 + \nu_2 \perp \mu$.
b) If $\nu_1 << \mu$ and $\nu_2 << \mu$, then $\nu_1 + \nu_2 << \mu$.
c) If $\nu_1 << \mu$, then $|\nu_1| << \mu$; and conversely.
d) If $\nu_1 << \mu$ and $\nu_2 \perp \mu$, then $\nu_1 \perp \nu_2$.
e) If $\nu_1 << \mu$ and $\nu_1 \perp \mu$, then $\nu_1 \equiv 0$.

3. Let $(X, \mathcal{A}, \mu)$ be a measure space, $\{E_k\}_{k=1}^n \subset \mathcal{A}$, and $\{c_k\}_{k=1}^n$ be a collection of real numbers. For $E \in \mathcal{A}$, define $\nu(E) = \sum_{k=1}^n c_k \cdot \mu(E \cap E_k)$. Show that $\nu << \mu$ and find $\frac{d\nu}{d\mu}$.

4. Let $\{\mu_n\}$ be a sequence of measures on a measurable space $(X, \mathcal{A})$ such that there is a constant C with $\mu_n(X) \leq C$ for all $n \geq 1$. Define $\nu : \mathcal{A} \to \mathbb{R}$ by

$$\nu(E) = \sum_{n=1}^{\infty} \frac{\mu_n(E)}{2^n}, \quad E \in \mathcal{A}.$$

Show that ν is a measure on $\mathcal{A}$ and that $\mu_n << \nu$ for each $n \geq 1$.

5. Consider the measure space $([0,1], \mathcal{F}|_{[0,1]}, m)$ and let ν be the counting measure on $\mathcal{F}|_{[0,1]}$. Show that

a) $m << \nu$, and
b) there is no function $f : [0,1] \to \mathbb{R}$ for which $m(E) = \int_E f d\nu$ for all $E \in \mathcal{F}|_{[0,1]}$.

6. Let $\{a_n\}_n$ be a fixed sequence of real numbers and $\{p_n\}_n$ be a sequence of positive real numbers. Define a set function $\nu : \mathcal{F} \to \mathbb{R}$ by

$$\nu(E) = \sum_{a_n \in E} p_n, \quad E \in \mathcal{F}.$$

a) Show that ν is a σ-finite measure on $\mathcal{F}$.
b) Find the Lebesgue decomposition of ν with respect to the Lebesgue measure on $\mathcal{F}$.

7. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2 - 6x + 5$, and define a set function $\nu : \mathcal{B} \to \mathbb{R}$ by

$$\nu(E) = \int_E f(x) dm, \quad E \in \mathcal{B}.$$

a) Show that ν is a σ-finite measure on $\mathcal{B}$.
b) Find the Hahn decomposition of $\mathbb{R}$ with respect to ν.
c) Find the Jordan decomposition of ν.
d) Find the Lebesgue decomposition of ν with respect to the Lebesgue measure on $\mathcal{F}$.

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Assignment #5; Due on Dec. 1, 2014

1. Let $E \subset \mathbb{R}$ be a bounded set (say $E \subset [a,b]$). If $\mathcal{V} = \{[\alpha,\beta] : a \leq \alpha < \beta \leq b,\ \alpha,\beta \in \mathbb{Q}\}$, show that $\mathcal{V}$ is a Vitali cover for E.

2. Let $f : [a,b] \to \mathbb{R}$ be a continuous function which is differentiable on (a,b), and let $E = \{x \in (a,b) : f'(x) > \alpha\}$. If $\mathcal{V} = \{[c,d] \subset (a,b) : f(d) - f(c) \geq \alpha(d-c)\}$, show that $\mathcal{V}$ is a Vitali cover for E.

3. a) Find a function $f : [a,b] \to \mathbb{R}$ such that $\overline{D}f(c) = \infty$, and $\underline{D}f(c) = 0$ for some $c \in (a,b)$.

b) Determine $\overline{D}f(0)$ and $\underline{D}f(0)$ for

$$f(x) = \begin{cases} x\sin(\frac{1}{x}) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

c) Determine $\overline{D}f(x)$ and $\underline{D}f(x)$ for the function $f : [0,1] \to \mathbb{R}$ given by $f(x) = \chi_{\mathbb{Q}}(x)$. Is f a function of bounded variation on $[0,1]$?

4. Consider functions $f, g : [-1,1] \to \mathbb{R}$ given by

$$f(x) = \begin{cases} x^2\cos(\frac{1}{x^2}) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

$$g(x) = \begin{cases} x^2\cos(\frac{1}{x}) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

a) Find $\overline{D}f(0)$, $\underline{D}f(0)$, $\overline{D}g(0)$ and $\underline{D}g(0)$.
b) Determine if f and g are of bounded variation on $[-1,1]$.

5. For any $E \subset \mathbb{R}$, let $BV(E)$ denote the collection of all $\mathbb{R}$-valued functions of bounded variation on E. Show that if $f, g \in BV(E)$ and $\alpha \in \mathbb{R}$, then $f+g$, $\alpha f \in BV(E)$. Is $fg \in BV(E)$? (Prove or provide a counterexample.)

6. Let $f \in BV([a,b])$. Show that there exists a countable family $\{\mathcal{P}_n\}_{n\geq 1}$ of partitions of $[a,b]$ such that

$$\lim_n V_a^b(f, \mathcal{P}_n) = T_a^b(f).$$

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Assignment #4; Due on Nov. 5, 2014

Solve any five of the following problems.

1. Let $E \in \mathcal{F}$ and $f : E \to \mathbb{R}$ be a function. f is a measurable function if and only if $f^{-1}(O) \in \mathcal{F}$, $\forall\, O \in \mathcal{B}$.

2. Let $\{f_n\}$ be a sequence of $\mathbb{R}$-valued measurable functions on $E \in \mathcal{F}$ and $f : E \to \mathbb{R}$ be measurable. $f_n \to f$ almost everywhere on E if and only if

$$m(\{x \in E : \limsup_n f_n(x) > \liminf_n f_n(x)\}) = 0.$$

3. Find a function $f : E \to \mathbb{R}$, $E \in \mathcal{F}$, such that $|f|$ is measurable whereas f is not.

4. Let $f : [a,b] \to \mathbb{R}$ be a bounded function. If f is Riemann-integrable, then f is Lebesgue-integrable and

$$\int_{[a,b]} f(x)dm = \int_a^b f(x)dx.$$

5. Let $E \in \mathcal{F}$ with $m(E) < \infty$ and $\{f_n\}$ be a sequence of $\mathbb{R}$-valued, bounded measurable functions on E such that $f_n \to f$ uniformly on E. Then

$$\int_E f_n dm \to \int_E f dm.$$

[Do not use BCT!]

6. The function $f(x) = \frac{1}{\sqrt{x}}$ is measurable on $[0,1]$. Calculate $\int_{[0,1]} \frac{1}{\sqrt{x}} dm$. [Notice: f is not Riemann-integrable on $[0,1]$.]

7. Find $\int_{[0,1]} f(x)\, dm$ if

$$f(x) = \begin{cases} x^2 & \text{if } x \in C \cap \mathbb{Q} \\ x & \text{if } x \in C \cap \mathbb{Q}^c \\ 1 & \text{if } x \in [0,1] \setminus C \end{cases}$$

where C is the standard Cantor Set.

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PROBLEMS

1. If $\mu_1, \mu_2, \ldots, \mu_n$ are measures on a measurable space $(X, \mathcal{A})$ and $\alpha_1, \alpha_2, \ldots, \alpha_n \in [0, \infty)$, then $\mu = \sum_{i=1}^n \alpha_i \mu_i$ is also a measure on $(X, \mathcal{A})$.

2. If $(X, \mathcal{A}, \mu)$ is a measure space and $E \in \mathcal{A}$, then the set function $\mu_E : \mathcal{A} \to [0, \infty]$, defined by $\mu_E(A) = \mu(E \cap A)$ for all $A \in \mathcal{A}$, is also a measure on $\mathcal{A}$.

3. Let μ be a finitely additive measure on a σ-algebra $\mathcal{A}$ of subsets of a set X. μ is a measure if and only if it is continuous from below.

4. Let μ be a finitely additive measure on a σ-algebra $\mathcal{A}$ of subsets of a set X with $\mu(X) < \infty$. μ is a measure if and only if it is continuous from above.

5. Let $(X, \mathcal{A}, \mu)$ be a measure space.

a) If $A, B \in \mathcal{A}$ and $\mu(A \triangle B) = 0$, then $\mu(A) = \mu(B)$.
b) Define: $A \sim B$ if and only if $\mu(A \triangle B) = 0$. Then "$\sim$" is an equivalence relation on $\mathcal{A}$.
c) For $A, B \in \mathcal{A}$, define $\rho(A, B) = \mu(A \triangle B)$. Then ρ is a metric on $\mathcal{A}/\sim$.
(Hence, $(\mathcal{A}/\sim\ ,\ \rho)$ is a metric space).

6. If an outer measure μ^* on a set $X \neq \emptyset$ is finitely additive, then it is a measure.

7. Let $X \neq \emptyset$ and $\kappa = \{\emptyset,\ X,\ \text{one-point sets}\}$. Define $\lambda(X) = \infty$, $\lambda(\emptyset) = 0$, and $\lambda(E) = 1$ if $\emptyset \neq E \subsetneq X$. Describe the outer measure μ_λ^*.

8. Let X be an uncountable set and $\kappa = \{\emptyset,\ X,\ \text{one-point sets}\}$. Define $\lambda(X) = 1$ and $\lambda(E) = 0$ if $E \subsetneq X$. Describe the outer measure μ_λ^*.

9. Let $X \neq \emptyset$, κ be a covering class for X and λ be a set function on $\mathcal{P}(X)$. If $E \in \kappa$, then $\mu_\lambda^*(E) \leq \lambda(E)$. Give an example where strict inequality occurs.

10. Let $X \neq \emptyset$, κ be a covering class for X, λ be a set function on $\mathcal{P}(X)$ and μ_λ^* be the outer measure induced by κ and λ.

a) If κ is a σ-algebra and λ is a measure, then $\mu_\lambda^*(E) = \lambda(E)$ for all $E \in \kappa$.
b) If κ is a σ-algebra and λ is a measure, then every set in κ is μ_λ^*-measurable.

[The measure μ_λ given by μ_λ^* is called an **extension** of λ.]

11. Let $X \neq \emptyset$, $\kappa = \{\emptyset, X\}$ and λ be a set function on $\mathcal{P}(X)$ given by $\lambda(E) = 0$ if $E = \emptyset$ and $\lambda(E) = 0$ if $E = X$. Determine the outer measure μ_λ^* and describe the σ-algebra of μ_λ^*-measurable sets. The same question when $X = [0, 1]$.

12. Let $X = \mathbb{R}$, $\kappa = \mathcal{P}(\mathbb{R})$ and define $\lambda(E) = |\{k \in \mathbb{Z} :\ k \in E\}|$. Determine the outer measure μ_λ^* and describe the σ-algebra of μ_λ^*-measurable sets.

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Assignment #2; Due on Oct. 1, 2014

Solve any five of the following problems.

1. Show that $m^*(A) = m^*(\alpha + A)$ for all $A \subset \mathbb{R}$, $\alpha \in \mathbb{R}$.

2. Calculate $m^*(A)$ if

(i) A is a singleton or a finite set,
(ii) $A = \mathbb{Q} \cap [0, 1]$.

3. Show that

(i) For any $a, b \in \mathbb{R}$, $a < b$, the sets (a, b), $[a, b)$, $(a, b]$, and $[a, b]$ are measurable.
(ii) $\mathbb{Q}$ and $\mathbb{Q}^c$ are measurable, and calculate $m(\mathbb{Q}^c \cap [0, 1])$.

4.Let $E \subset \mathbb{R}$ with $m^*(E) > 0$. Show that there exists a bounded subset of E that also has positive outer measure.

5. Show that if $A, B \in \mathcal{F}$, then $m(A \cup B) + m(A \cap B) = m(A) + m(B)$.

6. Prove (i) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (v) in Theorem.2; that is, TFAE:

(i) $E \in \mathcal{F}$
(iii) For all $\varepsilon > 0$, there exists C closed such that $C \subset E$ and $m^*(E \setminus C) < \varepsilon$.
(v) There exists $F \subset \mathcal{F}_\sigma$ with $F \subset E$ such that $m^*(E \setminus F) = 0$.

[Hint: Use the fact that (i) $\Leftrightarrow$ (ii).]

7. Prove that $A \in \mathcal{F}$ if and only if for any $\epsilon > 0$ there is a closed set C and an open set O with $C \subset A \subset O$ such that $m^*(O \setminus C) < \epsilon$.

8. Let $E \subset \mathbb{R}$ have finite outer measure. Prove that $E \in \mathcal{F}$ if and only if for each bounded interval (a, b),

$$b - a = m^*((a, b) \cap E) + m^*((a, b) \setminus E).$$

9. Given a collection of subsets $\{E_n\}_n^\infty$ of a set X, similarly to the way we defined the $\limsup a_n$ and $\liminf a_n$ of a sequence, we define

$$\limsup_n A_n = \cap_{n=1}^\infty \cup_{k=n}^\infty A_k, \text{ and}$$
$$\liminf_n A_n = \cup_{n=1}^\infty \cap_{k=n}^\infty A_k.$$

Show that for $\{A_n\}_n \subset \mathcal{F}$.

(i) $m(\liminf_n A_n) \leq \liminf_n m(A_n)$.
(ii) If $m(\cup_{n=1}^\infty A_n) < \infty$, then $m(\limsup_n m(A_n) \leq m(\limsup_n A_n)$.

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Assignment-I (Due on Sept. 17, 2014)

1. Recall that for $S \subset [a,b]$ and $\mathcal{P} = \{(x_i, x_{i+1})\}$ any finite partition of $[a,b]$ by open intervals, we define

$$J_i(S) = \{\sum l(I_i) : I_i \subset S,\ I_i \in \mathcal{P} \text{ contains interior points of } S\},$$

$$J_o(S) = \{\sum l(I_i) : I_i \in \mathcal{P} \text{ contains points of } S \cup \partial S\},$$

and let $c_i(S) = \sup_{\mathcal{P}} J_i(S)$ and $c_o(S) = \inf_{\mathcal{P}} J_o(S)$. Then, if $c_i(S)$ and $c_o(S)$ exist and are equal, we define $c(S) =$ **content of** $S = c_i(S)$ and call S *Jordan-measurable.* Prove that

(i) $c(E) \geq 0$ (non-negativity).

(ii) If $A \subset B$ are Jordan-measurable sets, then $c(A) \leq c(B)$ (monotonicity).

(iii) If E and F are disjoint Jordan-measurable sets, $c(E \cup F) = c(E) + c(F)$ (finite additivity); if E and F are not necessarily disjoint, $c(E \cup F) \leq c(E) + c(F)$ (subadditivity).

(iv) If A is Jordan measurable and α is a real number, then $\alpha + A$ is also Jordan measurable and $c(A) = c(\alpha + A)$ (translation invariance).

2. Provide simple justifications that any $\mathcal{F}_\sigma$, $\mathcal{G}_\delta$, $\mathcal{F}_{\sigma\delta}$ and $\mathcal{G}_{\delta\sigma}$-set is a Borel set.

3. (i) Show that $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Q}^c$, $\mathcal{C}$ are Borel sets, where $\mathcal{C}$ is the Cantor set. Which one of the classes $\mathcal{F}_\sigma$, $\mathcal{G}_\delta$, $\mathcal{F}_{\sigma\delta}, \ldots$ each of these sets belong?

(ii) Prove that translate of any $\mathcal{F}_\sigma$-set ($\mathcal{G}_\delta$-set) is a $\mathcal{F}_\sigma$-set ($\mathcal{G}_\delta$-set).