Recurrence theorems for admissible superadditive processes

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1109^{th} AMS Meeting at University of Alabama, Huntsville

March 27, 2015

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For every $A \in \Sigma$ *and for every* $\epsilon > 0$, *there exists a relatively dense subset* N *of* $\mathbb{N}$ *such that* $\mu(A \cap T^{-n}A) \geq \mu(A)^2 - \epsilon$ *for every* $n \in N$.

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The return of a point to its original neighborhood (guaranteed by PRT) takes place in a fairly regular fashion. Hence, the sequence of the return times has bounded gaps, i.e., a syndetic set.

Fürstenberg's Multiple Recurrence Theorem (FMRT)

$\forall A \in \Sigma$ *with* $\mu(A) > 0$ *and* $k \geq 1$,

$$\liminf_N \frac{1}{N} \sum_{n=0}^{N-1} \mu(A \cap T^{-k}A \cap T^{-2k}A \cap \ldots \cap T^{-nk}A) > 0,$$

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Theorem.A [Furstenberg, 1977]

If T is weakly mixing, then $\forall k \geq 1$ and $\{f_j\}_{j=1}^k \subset L_\infty$,
$\frac{1}{N}\sum_{n=1}^N f_1(T^n x) f_2(T^{2n}x)\ldots f_k(T^{kn}x) \to \int f_1 \int f_2 \ldots \int f_k$ in L_2.

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Theorem.B [Leibman, 2005]

If T is an impt, then $\forall k \geq 1$ integer polynomials $p_1, \ p_2, \ \ldots, \ p_k$ with $p_i(0) = 0$, and $\{f_j\}_{j=1}^{k} \subset L_\infty$, $\frac{1}{N}\sum_{n=1}^{N} f_1(T^{p_1(n)}x)\ldots f_k(T^{p_k(n)}x)$ converges in L_2.

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If T is totally ergodic and p_i's are rationally independent, then the L_2-limit is $\int f_1 \ldots \int f_k$. [Host and Kra, 2005]

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Theorem.C [Tao, 2008]

If $T_1, \ T_2, \ldots, \ T_k$ are commuting impt's, then for all $\{f_j\}_{j=1}^k \subset L_\infty(X)$, $\frac{1}{N}\sum_{n=1}^N f_1(T_1^n x) f_2(T_2^n x)\ldots f_k(T_k^n x)$ converges in L_2.

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Theorem D. [Frantzikinakis and Kra, 2005]

If T is an impt, then $\forall k \geq 1$, rationally independent integer polynomials $p_1, \ldots, p_k$ with $p_i(0) = 0$, and $A \in \Sigma$ with $\mu(A) > 0$, the set $\{n \in \mathbb{Z} : \mu(A \cap T^{-p_1(n)}A \cap \cdots \cap T^{-p_k(n)}A) \geq \mu(A)^{k+1} - \epsilon\}$ is syndetic.

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Theorem E. [Bergelson, Host and Kra, 2005]

If T is an impt, then for all $A \in \Sigma$ with $\mu(A) > 0$, the set $\{n \in \mathbb{Z} : \mu(A \cap T^{-n}A \cap T^{-2n}A \cap T^{-3n}A) \geq \mu(A)^4 - \epsilon\}$ is syndetic.

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Theorem F. [Bergelson, 2000]

Let T be an impt. If $f \in L_\infty$, $f \geq 0$, then

$$\lim_{N\to\infty} \frac{1}{N^2} \sum_{m,n=1}^{N} \int [f(T^m f)(T^n f)(T^{n+m} f)] d\mu \geq \left(\int f d\mu\right)^4.$$

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Hence, for any $\epsilon > 0$, and $A \in \Sigma$ with $\mu(A) > 0$, the set

$$\{(m,n) \in \mathbb{Z}^2 : \mu(A \cap T^{-m}A \cap T^{-n}A \cap T^{-(m+n)}A) \geq \mu(A)^4 - \epsilon\}$$

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Motivation

Let $\{A_n\}$ be measurable subsets of $\mathbb{R}$ with $A_n \uparrow A$ and $\sum m(A_n \backslash A_{n-1}) < \infty$. Then for any ergodic mpt T on $\mathbb{R}$, define the "generalized return-time sequence" by

$$n_k = \inf\{r \in \mathbb{Z} : T^r(x) \in A_k, r \geq n_{k-1}\}.$$

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Let $\{A_n\}$ be measurable subsets of a compact interval X of $\mathbb{R}$ with $A_n \uparrow A$ and $\sum m(A_n \backslash A_{n-1}) < \infty$, and T be a impt on X. Then

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b) $\forall \epsilon > 0$, *the set* $\{n \in \mathbb{Z} : m(A \cap T^{-n} A_n) > m(A)^2 - \epsilon\}$ *is syndetic.*

Observe: If $f_k := T^k \chi_{A_k}$, then the family $\{f_k\}$ defines a T-admissible process. Hence (a) and (b) above can be rewritten in terms of such processes.

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4. Any strongly bounded T-superadditive process $F \subset L_1$ admits an exact dominant [Akcoglu and Sucheston, 1978].

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If T be an impt on X, then $\forall k \geq 1$, non-negative integer polynomials $p_1, \ldots, p_k$ with $p_i(0) = 0$, and strongly bounded T-admissible processes $\{f_n^{(j)}\}_{j=1}^k \subset L_\infty$, then $\frac{1}{N}\sum_{n=1}^N f_{p_1(n)}^{(1)} f_{p_2(n)}^{(2)} \ldots f_{p_k(n)}^{(k)}$ converges in L_2.

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$$\mu(A \cap T^{-n}A_n \cap T^{-2n}A_{2n} \cap \cdots \cap T^{-kn}A_{kn}) > 0.$$

Hence, there exists a time n at which all the sets $\{T^{-jn}A_{jn}\}_{j=1}^k$ meet A simultaneously.

Theorem.2

Let $T_1,\ T_2, \ldots,\ T_k$ be commuting impt's. If $F^{(j)} = \{f_n^{(j)}\} \subset L_\infty$ are strongly bounded T_j-admissible processes, $1 \leq j \leq k$, then $\frac{1}{N}\sum_{n=1}^{N} f_n^{(1)} f_n^{(2)} \ldots f_n^{(k)}$ converges in L_2.

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Theorem.3

If T is an impt and $F = \{f_i\} \subset L_\infty(X)$ is a non-negative strongly bounded T-admissible process with exact dominant δ, then

$$\lim_{N\to\infty} \frac{1}{N^2} \sum_{m,n=1}^{N} \int \delta f_m f_n f_{n+m} d\mu \geq \left(\int \delta d\mu\right)^4.$$

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Hence, for any $\epsilon > 0$, and $f_i^{(j)} = T_j^i \chi_{A_i}$, $1 \leq j \leq k$, and $\delta = \chi_A$, where $A_i \uparrow A$, the set

$$\{(m,n) \in \mathbb{Z}^2 : \mu(A \cap T^{-m} A_m \cap T^{-n} A_n \cap T^{-(m+n)} A_{m+n}) \geq \mu(A)^4 - \epsilon\}$$

is syndetic.

1. Any dominated T-superadditive process $F = \{F_n\}$ can be decomposed as $F = G - H$, where $G = \{G_n\}$ is additive and $H = \{H_n\}$ is subadditive (i.e., $H_{m+n} \leq H_m + T^m H_n$).

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Proposition.

Let F be a strongly bounded T-admissible process with subadditive part $\{h_i\}$. Then $h_i \geq 0$ for all $i \geq 0$, and $\frac{1}{n} \| \sum_{i=0}^{n-1} h_i \|_2 \to 0$.

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(Sketch of the proof.) It is enough to prove that

$$\|\frac{1}{N}\sum_{n=1}^{N} f_n^{(1)} \dots f_{kn}^{(k)} - \frac{1}{N}\sum_{n=1}^{N} T^n\delta^{(1)} \dots T^{kn}\delta^{(k)}\|_2 \to 0,$$

where $\delta^{(j)}$ is the exact dominant for $F^{(j)}$.

- Each $F^{(j)}$ has an exact dominant $\delta^{(j)}$, and hence, for each $j = 1, \ldots, k$, $f_n^{(j)} = T^n \delta^{(j)} - h_n^{(j)}$.

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is bounded by $(2^k - 1)$-terms $\| \frac{1}{N} \sum_{n=1}^{N} (\ldots h_{un}^{(u)} T^{vn} \delta^{(v)} \ldots) \|_2$, for some $1 \leq u, v \leq k$, but at least one of the factors is $h_{jn}^{(j)}$.

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- Hence, it remains to show that all these $(2^k - 1)$-terms converge to zero.

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It is an average of the form $\frac{1}{N}\sum_{i=1}^{N} r_i(\omega)$, where $R = \{r_i\} = \{h_i^{(1)} h_{2i}^{(2)} T^{3i}\delta^{(3)} \dots T^{ki}\delta^{(k)}\}$ is a process on L_1-space of the product space $(\prod_1^k X, \prod_1^k \Sigma, \prod_1^k \mu)$ with $S = T \times T^2 \times T^3 \times \cdots \times T^k$.

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All the other $(2^k - 1)$-terms of this kind are handled similarly with L_2-limit zero. □

Question: Do Theorem.1 - Theorem.3 also valid for arbitrary superadditive processes?

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Thank You!