Good Modulating Sequences for the Ergodic Hilbert Transform

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Set up

Let (X, Σ, μ) be a probability space and $T : X \to X$ be an i.m.p.t. For a function f the **ergodic Hilbert transform (eHt) of** f is

$$Hf(x) := \lim_n \sum_{k=-n}^{n}{}' \frac{T^k f(x)}{k}, \quad \text{if the limit exists.}$$

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Given a sequence $\mathbf{a} = \{a_k\}_{k=-\infty}^{\infty} \subset \mathbb{C}$, the **modulated ergodic Hilbert transform of** f **by** $\mathbf{a}$ is defined as

$$H_{\mathbf{a}} f(x) := \lim_{n} \sum_{-n}^{n}{}' \frac{a_k T^k f(x)}{k}.$$

Recall: For $f \in L_p$, modulated ergodic averages by $\mathbf{a}$ is defined as

$$A_n(\mathbf{a}, f)(x) := \frac{1}{n} \sum_{k=0}^{n-1} a_k T^k f(x).$$

If $\lim_n A_n(\mathbf{a}, f)$ exists a.e. for every $f \in L_p(X)$, $\mathbf{a}$ is called L_p-**good in the system** (X, Σ, μ); if this holds in every dynamical system, then $\mathbf{a}$ is called a **universally** L_p-**good sequence**.

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- W_p-sequences (L_q-good) [Lin-Olsen-Tempelman, 1999].

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Cohen-Lin, 2003

Let **a** be a sequence of bounded variation. If $f \in L_p$, $1 < p < \infty$, satisfies

$$\sup_n \| \frac{1}{n^{1-\beta}} \sum_{k=1}^{n} T^k f \|_p = K < \infty \text{ for some } 0 < \beta \leq 1,$$

then the modulated one-sided eHt exists.

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then the modulated one-sided eHt exists for all $f \in L_p$, $2 < p < \infty$. In particular, if $\mathbf{a} \in W_p$, $1 < p < \infty$ satisfying this rate condition, then the modulated one-sided eHt exists for all $f \in L_q$.

Modulated Ergodic Hilbert Transform

Definition

If (X, Σ, μ, T) is an i.m.p.s., **a** is called L_p-**good for the eHt in** (X, Σ, μ) if the modulated eHt exists a.e. for every $f \in L_p(X)$.

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- There are bounded Besicovitch sequences that are not good for the eHt.
- If $\mathbf{a} = \{\hat{f}_n\}$ as in 1-3 above, then it satisfies $\sum_{-n}^{n} |\hat{f}_n| = O(n^\beta)$ for some $0 < \beta < 1$.

Universally Good Sequences for the eHt

For $1 < \alpha \leq 2$ define

$$M_\alpha = \{\mathbf{a} = \{a_k\}_{-\infty}^{\infty} \subset \mathbb{C} : \sum_{k=-n}^{n} |a_k| = O(\frac{n^{\alpha-1}}{\log^\alpha n})\}.$$

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- If $\alpha > \beta + 1,\ 0 < \beta < 1$, then, for n is large enough, $n^\beta \leq \frac{n^{\alpha-1}}{\log^\alpha n}$; hence, any sequence satisfying the condition $\sum_{-n}^{n} |a_n| = O(n^\beta)$ belongs to M_α.

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- If $\mathbf{a} \in M_\alpha$, then $\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} a_k T^k f = 0$ a.e. for all $f \in L_\infty$.

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- $\exists\ \mathbf{a} \in A_\alpha$ that does not belong to any M_α.
- If $\mathbf{a} \in A_\alpha,\ 1 < \alpha < 3/2$, then $\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} a_k T^k f = 0$ a.e. for all $f \in L_\infty$.

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Theorem.2

If $\mathbf{a} \in A_\alpha,\ 1 < \alpha \leq 2$, is a sequence L_2-good for the ergodic averages, then it is universally L_p-good for the eHt, $2 \leq p < \infty$.

Proof of Theorem.1 (Sketch)

- By Abel's partial summation, for $f \in L_1$, $\sum_{-n}^{\prime n} \frac{a_k T^k f}{k} = R_n + \frac{1}{n}(S_n - S_{-n})$, where $R_n = \sum_1^{n-1} \frac{S_k - S_{-k}}{k(k+1)}$ and $S_{\mp j} = \sum_1^j a_{\mp i} T^{\mp i} f$. We have weak (1,1) maximal inequality for $\frac{1}{n} S_{\mp j}$.

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- Markov's inequality $\Rightarrow$ $\mu(\{\sup_n |R_n| > \lambda\}) \leq \frac{C}{\lambda}\|f\|_1$; and hence, we have the weak (1,1) maximal inequality for $H_{\mathbf{a}} f$.
- For any $f \in L_\infty$, $H_{\mathbf{a}} f$ exists. Hence, the Banach Principle implies the assertion. □

Bounded Besicovitch Sequences

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Theorem (Lacey-Terwilleger, 2008)

If $f \in L_p$, $1 < p < \infty$, then there is a set $X_f \subset X$ of probability one such that for all $x \in X_f$

$$\lim_n \sum_{k=-n}^{n}{}' \frac{\lambda^k T^k f(x)}{k} \text{ exists for all } |\lambda| = 1.$$

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Let $\mathscr{W}$ denote the class of sequences induced by bounded trigonometric polynomials. Hence, if $\mathbf{w} \in \mathcal{W}$ then it is universally L_p-good, $1 < p < \infty$.

Definition

For $1 < \alpha \leq 2$, let $AB_\alpha = \{\mathbf{a} \in l_\infty : \exists\ \mathbf{w} \in \mathscr{W} \ni \mathbf{a} - \mathbf{w} \in A_\alpha\}$.

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Note: If $\mathbf{a} \in l_\infty$ with $\sum_{k=-n}^{n} |a_k - w_k| = O(n^\beta)$ for some $\mathbf{w} \in \mathscr{W}$ and $0 < \beta < 1$ (which are good in L_1 for the eHt [Ç, 2009]), then $\mathbf{a} - \mathbf{w} \in A_\alpha,\ 1 < \alpha \leq 2$.

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Non-symmetric sequences case:

Let $\mathbf{a} \in AB_\alpha$ be a non-symmetric sequence ($\mathbf{a}$ is called **symmetric** if $a_k = a_{-k},\ k \geq 1$) and let $\mathbf{w} = \{w_k\}$ be a trigonometric polynomial such that $\mathbf{a} - \mathbf{w} \in AB_\alpha$. Now, $a_k = a_k - w_k + w_k$, and $\{\sum\nolimits_{-n}^{\prime n} \frac{(a_k - w_k) T^k f}{k}\}_n$ converges a.e. by Theorem.2. This fact, combined with Lacey-Terwilleger Theorem, gives

Corollary

If $\mathbf{a} \in AB_\alpha$ is non-symmetric, then it is universally L_p-good for the eHt, $2 \leq p < \infty$.

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Symmetric sequences case:

If $\mathbf{w} = \{\lambda^{|k|}\}$, $|\lambda| = 1$, is a non-constant (symmetric) sequence, then there exists an irrational rotation $(\mathbb{T}, T)$ and $f \in L_1(X)$ such that $\sum_{-n}^{\prime n} \frac{\lambda^{|k|} T^k f}{k}$ diverges a.e.

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If (X, Σ, μ, T) is ergodic, then $L_2(X) = \kappa \oplus \kappa^\perp$, where κ is the Kronecker factor. Hence, for a bounded Besicovitch sequence $\mathbf{a}$ and an i.e.m.p. system with $\{f \in L_2 : Tf = f\} \subset \kappa$ properly, if $\lambda \in \sigma(\mathbf{a}) \cap \sigma(T)$, then $\lim_n \sum_{-n}^{\prime n} \frac{\lambda^{|k|} T^k f}{k}$ need not exist a.e.

Let $\mathscr{F}$ denote the class of weakly mixing i.m.p. systems. Any $T \in \mathscr{F}$ has continuous spectrum; hence, $\kappa = \{f \in L_2 : Tf = f\}$. Therefore, if T is weakly mixing, $\lim_n \sum_{-n}^{\prime n} \frac{\lambda^{|k|} T^k f}{k}$ exists a.e. for any $f \in \kappa$. Furthermore, Let $\mathscr{F}$ denote the class of weakly mixing i.m.p. systems. Any $T \in \mathscr{F}$ has continuous spectrum; hence, $\kappa = \{f \in L_2 : Tf = f\}$. Therefore, if T is weakly mixing, $\lim_n \sum_{-n}^{\prime n} \frac{\lambda^{|k|} T^k f}{k}$ exists a.e. for any $f \in \kappa$. Furthermore,

Theorem.3

If $\mathbf{a} \in AB_\alpha$ is a symmetric sequence, then it is universally L_2-good for the eHt in the class $\mathscr{F}$.

Let $\mathscr{L}$ denote the class of i.m.p. systems having Lebesgue spectrum. Hence, the spectral measure of any non-constant $f \in L_2$ is absolutely continuous w.r.t. the Lebesgue measure.

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Theorem.4

Let $T \in \mathscr{L}$ and $\mathbf{a}$ be a symmetric bounded sequence which is universally L_2-good (for the ergodic averages). Then $\mathbf{a}$ is universally L_2-good for the eHt in the class $\mathscr{L}$.

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Remark. There are uniform sequences which are not good for the eHt.