Good Modulating Sequences for the Ergodic Hilbert Transform

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Set up

Let \((X, \Sigma, \mu)\) be a probability space and \(T : X \rightarrow X\) be an i.m.p.t. For a function \(f\) the **ergodic Hilbert transform (eHt)** of \(f\) is

\[
Hf(x) := \lim_{n \to \infty} \sum_{k=-n}^{n} \frac{T^k f(x)}{k}, \quad \text{if the limit exists.}
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Given a sequence \(a = \{a_k\}_{k=-\infty}^{\infty} \subset \mathbb{C}\), the **modulated ergodic Hilbert transform of** \(f\) by \(a\) is defined as

\[
H_a f(x) := \lim_{n \to \infty} \sum_{k=-n}^{n} \frac{a_k T^k f(x)}{k}.
\]
Recall: For $f \in L_p$, modulated ergodic averages by $a$ is defined as

$$A_n(a, f)(x) := \frac{1}{n} \sum_{k=0}^{n-1} a_k T^k f(x).$$

If $\lim_n A_n(a, f)$ exists a.e. for every $f \in L_p(X)$, $a$ is called $L_p$-good in the system $(X, \Sigma, \mu)$; if this holds in every dynamical system, then $a$ is called a universally $L_p$-good sequence.
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If \( \lim_n A_n(a, f) \) exists a.e. for every \( f \in L_p(X) \), \( a \) is called \( L_p \)-good in the system \((X, \Sigma, \mu)\); if this holds in every dynamical system, then \( a \) is called a universally \( L_p \)-good sequence.

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- $W_p$-sequences ($L_q$-good) [Lin-Olsen-Tempelman, 1999].
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Modulated one-sided eHt of $f \in L_p$ by a complex sequence $a$ is defined as $\sum_{k=1}^{\infty} \frac{a_k T^k f(x)}{k}$, if the series converges.
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**Cohen-Lin, 2003**

Let $a$ be a sequence of bounded variation. If $f \in L_p$, $1 < p < \infty$, satisfies

$$\sup_n \left\| \frac{1}{n^{1-\beta}} \sum_{k=1}^{n} T^k f \right\|_p = K < \infty$$

for some $0 < \beta \leq 1$,

then the modulated one-sided eHt exists.
Cohen-Jones-Lin, 2004

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then the modulated one-sided eHt exists for all \( f \in L_p, 2 < p < \infty \). In particular, if \( a \in W_p, 1 < p < \infty \) satisfying this rate condition, then the modulated one-sided eHt exists for all \( f \in L_q \).
Definition

If $(X, \Sigma, \mu, T)$ is an i.m.p.s., $a$ is called $L_p$-good for the eHt in $(X, \Sigma, \mu)$ if the modulated eHt exists a.e. for every $f \in L_p(X)$.

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Modulated Ergodic Hilbert Transform

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**Definition**

A sequence \(a\) is universally \(L_p\)-good for the eHt in the class \(\mathcal{F}\) if it is \(L_p\)-good for the eHt in every dynamical system in \(\mathcal{F}\).
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Universally Good Sequences for the eHt
Bounded Besicovitch Sequences

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- There exists \( f \in L_1[0, 2\pi] \) such that \( \{\hat{f}_n\} \) is not good for the eHt.
- There are bounded Besicovitch sequences that are not good for the eHt.
- If \( a = \{\hat{f}_n\} \) as in 1-3 above, then it satisfies \( \sum_{-n}^{n} |\hat{f}_n| = O(n^\beta) \) for some \( 0 < \beta < 1 \).
For $1 < \alpha \leq 2$ define

$$M_\alpha = \{ a = \{ a_k \}_{-\infty}^{\infty} \subset \mathbb{C} : \sum_{k=-n}^{n} |a_k| = O\left( \frac{n^{\alpha-1}}{\log^{\alpha} n} \right) \}.$$
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If $\alpha > \beta + 1$, $0 < \beta < 1$, then, for $n$ is large enough, $n^\beta \leq \frac{n^{\alpha-1}}{\log\alpha n}$; hence, any sequence satisfying the condition $\sum_{-n}^{n} |a_n| = O(n^\beta)$ belongs to $M_\alpha$. 
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- If $a \in M_\alpha$, then $\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} a_k T^k f = 0$ a.e. for all $f \in L_\infty$. 

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A larger class of complex sequences is

For $1 < \alpha \leq 2$ define

$$A_\alpha = \{a = \{a_k\}_{-\infty}^{\infty} \subset \mathbb{C} : \sup_{n \geq 1} \max_{|z|=1} \frac{\log_\alpha n}{n^{\alpha-1}} \left| \sum_{k=-n}^{n} a_k z^k \right| = C_a < \infty \}.$$
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- $\exists a \in A_{\alpha}$ that does not belong to any $M_{\alpha}$.
- If $a \in A_{\alpha}$, $1 < \alpha < 3/2$, then $\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} a_k T^k f = 0$ a.e. for all $f \in L_{\infty}$. 
Theorem 1

If \( a \in M_\alpha \), \( 1 < \alpha \leq 2 \), then it is universally \( L_1 \)-good for the eHt.
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When \( 1 < \alpha < 3/2 \), sequences in \( A_\alpha \) also belong to a \( M_{\alpha'} \) (for a larger \( \alpha' \)). Hence, any \( a \in A_\alpha \) is also universally good for the eHt if \( 1 < \alpha < 3/2 \). For \( 3/2 \leq \alpha \leq 2 \) we need different arguments.
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Theorem 2

If \( a \in A_\alpha \), \( 1 < \alpha \leq 2 \), is a sequence \( L_2 \)-good for the ergodic averages, then it is universally \( L_p \)-good for the eHt, \( 2 \leq p < \infty \).
Proof of Theorem 1 (Sketch)

- By Abel’s partial summation, for \( f \in L_1 \),

\[
\sum_{-n}^{n} a_k T^k f = R_n + \frac{1}{n} (S_n - S_{-n}),
\]

where \( R_n = \sum_{1}^{n-1} \frac{S_k - S_{-k}}{k(k+1)} \)

and \( S_{\mp j} = \sum_{1}^{j} a_{\mp i} T^{\mp i} f \). We have weak \((1,1)\) maximal inequality for \( \frac{1}{n} S_{\mp j} \).
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- Since \( f \in L_1 \) and \( a \in M_\alpha \), it follows that
  \[
  \int |R_n| \leq \|f\|_1 \sum_{2}^{n} \frac{C'}{k^{3-\alpha} \log^\alpha k} \leq C \|f\|_1.
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- $h_n = \sum_1^n \frac{1}{k^2} (\sum_{-k}^k |a_j| T_j |f|) \uparrow h$ with $\int h_n \leq C \|f\|_1$; hence, by the MCT, $\int |R_n| \leq \int h_n \uparrow \int h \leq C \|f\|_1$. 
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- Markov’s inequality \( \Rightarrow \mu(\{ \sup_n |R_n| > \lambda \}) \leq \frac{C}{\lambda} \|f\|_1 \); and hence, we have the weak \((1,1)\) maximal inequality for \( H_a f \).
**Proof of Theorem.1** (Sketch)

- By Abel’s partial summation, for \( f \in L_1 \),
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- For any \( f \in L_\infty \), \( H_a f \) exists. Hence, the Banach Principle implies the assertion.
Bounded Besicovitch Sequences

Recall: There are bounded Besicovitch sequences that are not good for the eHt.
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**Theorem (Lacey-Terwilleger, 2008)**

If $f \in L_p$, $1 < p < \infty$, then there is a set $X_f \subset X$ of probability one such that for all $x \in X_f$

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\lim_{n} \sum_{k=-n}^{n} \frac{\lambda^k \mathcal{T}^k f(x)}{k}
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\lim_{n} \sum_{k=-n}^{n} \frac{\lambda^k T^k f(x)}{k} \quad \text{exists for all} \quad |\lambda| = 1.
\]

Let \( \mathcal{W} \) denote the class of sequences induced by bounded trigonometric polynomials. Hence, if \( w \in \mathcal{W} \) then it is universally \( L_p \)-good, \( 1 < p < \infty \).
Definition

For $1 < \alpha \leq 2$, let $AB_\alpha = \{a \in l_\infty : \exists w \in W \; \exists a - w \in A_\alpha\}$. 

Note: If $a \in l_\infty$ with $\sum_{n} \sum_{k} (a_k - w_k) T_k f_k$ converges a.e. by Theorem 2. This fact, combined with Lacey-Terwilleger Theorem, gives...
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Note: If $a \in l_\infty$ with $\sum_{k=-n}^{n} |a_k - w_k| = O(n^\beta)$ for some $w \in W$ and $0 < \beta < 1$ (which are good in $L_1$ for the eHt [Ç, 2009]), then $a - w \in A_\alpha$, $1 < \alpha \leq 2$. 
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Note: If $a \in l_\infty$ with $\sum_{k=-n}^{n} |a_k - w_k| = O(n^\beta)$ for some $w \in W$ and $0 < \beta < 1$ (which are good in $L_1$ for the eHt [Ç, 2009]), then $a - w \in A_\alpha$, $1 < \alpha \leq 2$.

Non-symmetric sequences case:

Let $a \in AB_\alpha$ be a non-symmetric sequence ($a$ is called symmetric if $a_k = a_{-k}$, $k \geq 1$) and let $w = \{w_k\}$ be a trigonometric polynomial such that $a - w \in AB_\alpha$. Now, $a_k = a_{k} - w_k + w_k$, and $\left\{ \sum_{n=1}^{n} \frac{(a_k - w_k)T_k}{k} \right\}_n$ converges a.e. by Theorem.2. This fact, combined with Lacey-Terwilleger Theorem, gives
Corollary

If $a \in AB_\alpha$ is non-symmetric, then it is universally $L_p$-good for the eHt, $2 \leq p < \infty$. 
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If \( a \in AB_\alpha \) is non-symmetric, then it is universally \( L_p \)-good for the eHt, \( 2 \leq p < \infty \).

Symmetric sequences case:

If \( w = \{ |\lambda|^k \} \), \( |\lambda| = 1 \), is a non-constant (symmetric) sequence, then there exists an irrational rotation \((\mathbb{T}, T)\) and \( f \in L_1(X) \) such that \( \sum_{-n}^{n} -\frac{\lambda^{|k|}T^kf}{k} \) diverges a.e.
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If \((X, \Sigma, \mu, T)\) is ergodic, then \( L_2(X) = \kappa \oplus \kappa^\perp \), where \( \kappa \) is the Kronecker factor. Hence, for a bounded Besicovitch sequence \( a \) and an i.e.m.p. system with \( \{f \in L_2 : Tf = f \} \subset \kappa \) properly, if \( \lambda \in \sigma(a) \cap \sigma(T) \), then \( \lim_n \sum_{-n}^{n} \lambda|k| T^k f \) need not exist a.e.
Let $\mathcal{F}$ denote the class of weakly mixing i.m.p. systems. Any $T \in \mathcal{F}$ has continuous spectrum; hence, $\kappa = \{ f \in L_2 : Tf = f \}$. Therefore, if $T$ is weakly mixing, $\lim_{n} \sum_{-n}^{n} \frac{\lambda |k| T^k f}{k}$ exists a.e. for any $f \in \kappa$. Furthermore,
Let $\mathcal{F}$ denote the class of weakly mixing i.m.p. systems. Any $T \in \mathcal{F}$ has continuous spectrum; hence, $\kappa = \{ f \in L_2 : Tf = f \}$. Therefore, if $T$ is weakly mixing, $\lim_n \sum_{-n}^{n} \frac{\lambda |k| T^k f}{k}$ exists a.e. for any $f \in \kappa$. Furthermore,

**Theorem 3**

If $a \in AB_\alpha$ is a symmetric sequence, then it is universally $L_2$-good for the eHt in the class $\mathcal{F}$. 
Let $\mathcal{L}$ denote the class of i.m.p. systems having Lebesgue spectrum. Hence, the spectral measure of any non-constant $f \in L_2$ is absolutely continuous w.r.t. the Lebesgue measure.
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**Theorem 4**

Let \( T \in \mathcal{L} \) and \( a \) be a symmetric bounded sequence which is universally \( L_2 \)-good (for the ergodic averages). Then \( a \) is universally \( L_2 \)-good for the eHt in the class \( \mathcal{L} \).
Let $\mathcal{L}$ denote the class of i.m.p. systems having Lebesgue spectrum. Hence, the spectral measure of any non-constant $f \in L_2$ is absolutely continuous w.r.t. the Lebesgue measure.

**Theorem 4**

Let $T \in \mathcal{L}$ and $a$ be a symmetric bounded sequence which is universally $L_2$-good (for the ergodic averages). Then $a$ is universally $L_2$-good for the eHt in the class $\mathcal{L}$.

**Remark.** There are uniform sequences which are not good for the eHt.