MATH 165
FALL 2006
FINAL EXAM

1. (24 pt) Evaluate the following limits if they exist.

$$\text{a)} \lim_{x\to 1} \frac{\ln(x) - x + 1}{(x-1)\sin(\pi x)} \qquad \text{b)} \lim_{x\to\infty} \sin\left(\sqrt{\frac{2x^2}{3x^2 - x + 1}}\right) \qquad \text{c)} \lim_{x\to 0}\left(\frac{1}{3x+1}\right)^{\frac{4}{x}}$$

$$\text{d)} \lim_{x\to\infty} \frac{x\tan^{-1}(x)}{e^{2x}+1}$$

2. (24 pt) Find the derivatives of the following functions (for the first part, assume $F'(x)$ is differentiable).

$$\text{a)}\ f(x) = F(x)^{F'(x)} \qquad \text{b)}\ g(x) = \sin(x)\cos(x)\tan(x)\sec(x)\csc(x)\cot(x)$$

$$\text{c)}\ h(x) = \frac{\ln(x^2 + \sin(2x))}{\sqrt{\tan(x) + 3x}} \qquad \text{d)}\ k(x) = \int_{g(x)}^{f(x)} (3t^2 + 1)dt$$

3. (15 pt) Evaluate the following integrals.

$$\text{a)} \int \frac{3x}{\sqrt{x}+1}\,dx \qquad \text{b)} \int_0^{\frac{\pi}{4}} \frac{\tan(x)\sec^2(x)}{\tan^2(x)+3}dx \qquad \text{c)} \int \sin^3(x)dx$$

4. (5 pt) Use the definition of the derivative to find the derivative of $f(x) = \frac{a}{bx+c}$; b, c not both zero.

5. (5 pt) Evaluate the limit

$$\lim_{n\to\infty} \sum_{i=1}^{n} (\frac{b}{n}) \cos(i(\frac{b}{n})).$$

6. (8 pt) A cone of radius 4 feet and height 8 feet (with circular base facing up) is being filled with water at a rate of 20 cubic feet per minute. How fast is the area of the circular surface of the water in the cone changing when the depth of the water is 6 feet? (The volume of a cone with base area A and height h is given by $V = \frac{1}{3}Ah$.)

7. (10 pt) Consider a rectangle inscribed inside a circle of radius R. Find the proportion of the area of the *largest* rectangle that can be put inside this circle to the area of the circle.

8. (10 pt) Sketch the graph of the curve $f(x) = x^{\frac{5}{3}} - 5x^{\frac{2}{3}} = x^{\frac{2}{3}}(x-5)$. The first two derivatives are given by

$$f'(x) = \frac{5}{3}x^{-\frac{1}{3}}(x-2)$$

and

$$f''(x) = \frac{10}{9}x^{-\frac{4}{3}}(x+1).$$

9. (4 pt) Find the tangent line to $\sin(x+y) + 1 = \cos(x-y)$ at $(0,0)$.

10. (5 pt) Consider the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$; $a, b > 0$. Set up an integral that gives the area inside this ellipse and then find the area (hint: the substitution $u = \frac{x}{a}$ might help you to change this into an integral that you can evaluate).