MATH 270
SPRING 2003
HOMEWORK 10

Due Friday May 2, 2003.

1. Let H be a subgroup of the group G. We say that H is a *normal* subgroup of G if for all $g \in G$ and $h \in H$, $g^{-1}hg \in H$. Prove the following.

a) (3 pt) If $\phi : G \longrightarrow H$ is a group homomorphism, then ker(ϕ)=$\{x \in G | \phi(x) = e_H\}$ is a normal subgroup of G.
b) (3 pt) Let A be an abelian group. Show that any subgroup of A is normal.
c) (3 pt) If A is an abelian subgroup of G, is A necessarily normal?

2. Let $\phi : G \longrightarrow H$ be a group homomorphism and let $x \in G$.

a) (5 pt) If the order of x is finite, say $|x| = n < \infty$, show that the order of $\phi(x)$ divides n.
b) (5 pt) If the order of x is infinite, does it follow that the order of $\phi(x)$ is infinite?
c) (5 pt) Show that the group homomorphism $\phi : G \longrightarrow H$ is one to one if and only if ker(ϕ)= e_G.

3. Let G be a cyclic group.

a) (3 pt) Show that if $H \subseteq G$ is a subgroup, then H is cyclic.
b) (3 pt) Show that if K is a group and $\phi : G \longrightarrow K$ is a homomorphism, then im(ϕ)=$\{\phi(x) | x \in G\}$ is cyclic.
c) (3 pt) Show that any cyclic group is abelian.
d) (3 pt) Show that any cyclic group is countable.
e) (3 pt) Are the groups $(\mathbb{R}, +)$ and $(\mathbb{Q}, +)$ cyclic? Why or why not?
f) (3 pt) Find all n for which the group S_n is cyclic.

4. (6 pt) Show that any group with more than one element has a non-identity abelian subgroup. Suppose that G is a group such that every *proper* subgroup is abelian (that is, if $H \subsetneq G$ then H is abelian). Is G necessarily abelian?

5. An isomorphism of groups $\phi : G \longrightarrow G$ (that is an isomorphism from G to itself) is called an *automorphism*.

a) (5 pt) Show that Aut(G), the set of all automorphisms of G, forms a group.
b) (5 pt) If ϕ is an automorphism of G and $g \in G$, show that the order of g is the same as the order of $\phi(g)$.
c) (5 pt) If $x \in G$ is a fixed element, show that the map defined by $\phi_x(g) = x^{-1}gx$ is an automorphism of G.
d) (5 pt) Show that the map defined by $f(x) = x^{-1}$ is an automorphism of G if and only if G is abelian.