MATH 270
SUMMER 2004
HOMEWORK 10

Due Friday July 30, 2004.

1. Let G and H be groups and $\phi : G \longrightarrow H$ a homomorphism.
   a) (3 pt) Show that $\phi(e_G) = e_H$.
   b) (3 pt) Show that if $x \in G$ and $|x| = n < \infty$ then $|\phi(x)|$ divides $|x|$.
   c) (3 pt) Show that ϕ is one to one if and only if $\ker(\phi) = e_G$.

2. Let G be a group and H a nonempty subset of G.
   a) (5 pt) Show that H is a subgroup of G if and only if $x, y \in H \implies xy^{-1} \in H$.
   b) (5 pt) Show that if G is finite, then H is a subgroup of G if and only if $x, y \in H \implies xy \in H$.

3. Let G be a group. An isomorphism $\phi : G \longrightarrow G$ is called an *automorphism*.
   a) (3 pt) Show that the set of automorphisms of G forms a group under function composition (this group is denoted $\mathrm{Aut}(G)$).
   b) (3 pt) Show that if $g \in G$ and $\phi \in \mathrm{Aut}(G)$ then $|g| = |\phi(g)|$.
   c) (3 pt) Show that for all $x \in G$ the map $\phi_x : G \longrightarrow G$ defined by $\phi_x(g) = x^{-1}gx$ is in $\mathrm{Aut}(G)$.

4. Let S_n be the symmetric group on n letters and D_m be the (dihedral) group of symmetries on the regular $m-$gon.
   a) (5 pt) Show that $|D_m| = 2m$.
   b) (5 pt) Show that D_m can be realized as a subgroup of S_m.
   c) (5 pt) In the group S_n find the inverse of the element $(a_1\ a_2\ a_3\ \cdots\ a_k)$, with $a_i \neq a_j$ if $i \neq j$ and $k \leq n$.
   d) (5 pt) Let p be a positive prime integer. Find the number of subgroups of S_p of order p.