

MATH 270
SUMMER 2007
HOMEWORK 10

Due Friday July 27, 2007.

1. Let G be a group. Prove the following.
 - a) (3 pt) If $x \in G$, then x^{-1} is unique.
 - b) (3 pt) If $x \in G$ then $(x^{-1})^{-1} = x$.
 - c) (3 pt) If $x, y \in G$ then $(xy)^{-1} = y^{-1}x^{-1}$.

2. Let G be a finite group with subgroup $H \subseteq G$ and $x \in G$ an element. We define the order of H (written $|H|$) to be the cardinality of the set H . Additionally, we define the order of x (written $|x|$) to be the smallest positive integer such that $x^n = e$ (and if no such positive integer exists, we say that $|x| = \infty$). Consider the set $\langle x \rangle = \{x^n | n \in \mathbb{Z}\}$.
 - a) (5 pt) Show that $\langle x \rangle$ is a subgroup of G .
 - b) (5 pt) Show that if $|x|$ is finite then $|x| = |\langle x \rangle|$.
 - c) (5 pt) Show that if $x^m = e$ and $x^n = e$ for some $n, m \in \mathbb{N}$ then $|x|$ divides $\gcd(n, m)$.
 - d) (5 pt) Show that if G is a finite group and $x \in G$, then $|x|$ is finite.

3. (5 pt) Show that if G is a group with the property that $x^2 = e$ for all $x \in G$, then G is abelian.

4. (3 pt) Let G and H be groups. We define $G \times H = \{(g, h) | g \in G, h \in H\}$ with a binary operation given by

$$(g_1, h_1)(g_2, h_2) = (g_1g_2, h_1h_2).$$

Show that this operation makes the set $G \times H$ into a group (called the direct product of G and H).