MATH 270
SPRING 2003
HOMEWORK 11

Due Friday May 16, 2003 at the start of the final exam…but early submissions encouraged.

1. Let $f(x)$ be a real-valued function. We say that $\lim_{x\to a} f(x) = L$ if for all $\epsilon > 0$ there exists $\delta > 0$ such that $0 < |x-a| < \delta \Longrightarrow |f(x) - L| < \epsilon$. Prove the following.

   a) (3 pt) If $\lim_{x\to a} f(x) = L$, $\lim_{x\to a} g(x) = M$, and c is a constant then $\lim_{x\to a}(f(x) + g(x)) = L + M$ and $\lim_{x\to a} cf(x) = cL$.

   b) (3 pt) Use part a) to show that the sum of two continuous function is continuous.

   c) (3 pt) Show that the set of real-valued continuous functions with domain $\mathbb{R}$ forms a group under addition.

2. A subset $I \subseteq \mathbb{R}$ is called *open* if for all $x \in I$ there is a $\delta > 0$ such that $(x-\delta, x+\delta) \subseteq I$. A set J is called *closed* if $\mathbb{R} \setminus J$ is open.

   a) (3 pt) Prove that an arbitrary union of open sets is open, is this true for intersections? (Hint: a previous problem set.)

   b) (3 pt) Prove that an arbitrary intersection of closed sets is closed, is this true for unions? (Hint: a previous problem set.)

   c) (3 pt) Give an example of a set that is neither open nor closed and an example of a set that is both open and closed.

3. (5 pt) Give an example of a bounded set of rational numbers that contains no least upper bound and no greatest lower bound (in the rationals).