MATH 270
SUMMER 2004
HOMEWORK 11

Due Thursday August 5, 2004.

1. Consider the set of rational numbers $\mathbb{Q}$ and fix a prime number $p > 0$ in $\mathbb{Z}$.

a) (5 pt) Show that any nonzero rational number can be written uniquely in the form $\frac{a}{b}p^n$ where $\gcd(a,p) = 1 = \gcd(b,p)$ and $n \in \mathbb{Z}$. (By "uniqueness" here I mean that if $\frac{a}{b}p^n = \frac{c}{d}p^m$ then $n = m$ and $\frac{a}{b} = \frac{c}{d}$.)

b) (5 pt) Show that if a, b, c, d are all integers that are not divisible by p, and $m, n \in \mathbb{Z}$, and $\frac{a}{b}p^n - \frac{c}{d}p^m \neq 0$, then $\frac{a}{b}p^n - \frac{c}{d}p^m = \frac{\alpha}{\beta}p^k$ where $\alpha, \beta \in \mathbb{Z}$ are not divisible by p and $k \geq \min(m,n)$.

2. Again consider the rational numbers $\mathbb{Q}$ and a fixed positive prime $p \in \mathbb{Z}$. We write each nonzero rational number as $\frac{a}{b}p^n$ with $\gcd(a,p) = 1 = \gcd(b,p)$ as in the previous problem. We define a function $d_p : \mathbb{Q} \times \mathbb{Q} \longrightarrow \mathbb{R}$ by

$$d_p(x,y) = \begin{cases} 0 & \text{if } x = y, \\ 2^{-n} & \text{if } x - y = \frac{a}{b}p^n \neq 0. \end{cases}$$

a) (5 pt) Show that d_p is a metric on $\mathbb{Q}$.

b) (5 pt) Describe the unit circle in $\mathbb{Q}$ using the metric (that is, describe the rational numbers that are all one unit away from 0).

3. Let $\{U_i\}_{i \in \Lambda}$ be a collection of open sets in a metric space and $\{X_j\}_{j \in \Gamma}$ be a collection of closed sets.

a) (3 pt) Show that $\bigcup_{i \in \Lambda} U_i$ is an open set.

b) (3 pt) Show that $\bigcap_{j \in \Gamma} X_j$ is a closed set.

c) (3 pt) Restrict yourself the the real numbers (with the standard metric), $\mathbb{R}$, and give an example to show that an arbitrary intersection of open sets need not be open.

d) (3 pt) Show that an arbitrary union of closed sets need not be closed.

e) (3 pt) Find all of the sets in $\mathbb{R}$ that are both closed and open (you may use the fact that any bounded subset of $\mathbb{R}$ has a greatest lower bound and a least upper bound).

4. Consider the standard metric on $\mathbb{R}$ ($d(x,y) = |x - y|$).

a) (5 pt) Show that the set of rational numbers $\mathbb{Q}$ is a dense subset of $\mathbb{R}$.

b) (5 pt) Give an example of a bounded subset of the rational numbers that has no least upper bound (in the rational numbers). What is its least upper bound in $\mathbb{R}$?