MATH 270
SUMMER 2007
HOMEWORK 12

Due Tuesday July 31, 2007.

1. Let G and H be groups and $\phi : G \longrightarrow H$ be a homomorphism.
   a) (3 pt) Show that $\phi(e_G) = e_H$.
   b) (3 pt) Show that if $x \in G$ with $|x| = n < \infty$ then $|\phi(x)|$ divides $|x|$.
   c) (3 pt) Show that ϕ is one to one if and only if $\ker(\phi) = e_G$.

2. Let G be a group. An isomorphism $\phi : G \longrightarrow G$ is called an *automorphism*. The set of all automorphisms is denoted by $\mathrm{Aut}(G)$.
   a) (5 pt) Show that $\mathrm{Aut}(G)$ forms a group under the operation of function composition.
   b) (5 pt) Show that if $g \in G$ and $\phi \in \mathrm{Aut}(G)$ then $|g| = |\phi(g)|$.
   c) (5 pt) Show that for all $x \in G$ the map $\phi_x : G \longrightarrow G$ defined by $\phi_x(g) = x^{-1}gx$ is an element of $\mathrm{Aut}(G)$.
   d) (5 pt) Show that the map $\psi : G \longrightarrow G$ defined by $\psi(g) = g^{-1}$ is an element of $\mathrm{Aut}(G)$ if and only if G is abelian.

3. (3 pt) For this problem, we denote the integers modulo n ($n \in \mathbb{N}$) by $\mathbb{Z}_n$. Show that if G is a cyclic group then

$$G \cong \begin{cases} \mathbb{Z}_n, & \text{if } |G| = n; \\ \mathbb{Z}, & \text{if } G \text{ is infinite.} \end{cases}$$

4. (5 pt) We have seen in class that the groups $(\mathbb{R}, +)$ and $(\mathbb{R}^+, \cdot)$ are isomorphic. Is the analogue for the rational numbers true? That is, prove or disprove that $(\mathbb{Q}, +)$ and $(\mathbb{Q}^+, \cdot)$ are isomorphic groups.