MATH 270
SUMMER 2007
HOMEWORK 13

Due Thursday August 2, 2007.

1. Let $\{U_i\}_{i\in\Lambda}$ be a collection of open sets in a metric space and $\{X_j\}_{j\in\Gamma}$ be a collection of closed sets.

a) (3 pt) Show that $\bigcup_{i\in\Lambda} U_i$ is an open set.
b) (3 pt) Show that $\bigcap_{j\in\Gamma} X_j$ is a closed set.
c) (3 pt) Show that the intersection of a *finite* number of open sets is open.
d) (3 pt) Show that the union of a *finite* number of closed sets is closed.
e) (3 pt) Restrict yourself the the real numbers (with the standard metric), $\mathbb{R}$, and give an example to show that an arbitrary intersection of open sets need not be open.
f) (3 pt) Show that an arbitrary union of closed sets need not be closed.

2. Consider the standard metric on $\mathbb{R}$ ($d(x,y) = |x-y|$).

a) (5 pt) Show that the set of rational numbers $\mathbb{Q}$ is a dense subset of $\mathbb{R}$.
b) (5 pt) Give an example of a bounded subset of the rational numbers that has no least upper bound (in the rational numbers). What is its least upper bound in $\mathbb{R}$?

3. Let $f(x)$ be a real valued function (with domain $\mathbb{R}$). We say that $\lim_{x\to a} f(x) = L$ if for all $\epsilon > 0$ there exists a $\delta > 0$ such that $0 < |x-a| < \delta \Longrightarrow |f(x) - L| < \epsilon$.

a) (5 pt) Show that if $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$ then $\lim_{x\to a}(f(x)+g(x)) = L + M$.
b) (5 pt) Show that if c is a constant and $\lim_{x\to a} f(x) = L$, then $\lim_{x\to a}(cf(x)) = cL$.
c) (5 pt) If X is an open subset of the real line $\mathbb{R}$, show that the sum of two functions continuous on X is continuous on X (recall that a function is continuous at a if $\lim_{x\to a} f(x) = f(a)$ and is continuous on an open set if it is continous at every point in the set).
d) (5 pt) Let $\mathfrak{C}_X$ be the functions that are continous on X. Let $X \subseteq Y \subseteq \mathbb{R}$. Show that $\mathfrak{C}_Y$ is a subgroup of $\mathfrak{C}_X$ (where the operation is addition of functions).