MATH 270
SPRING 2003
HOMEWORK 2

Due Monday January 27, 2003.

1. (3 pt) Let x and y be real numbers. Prove the following properties of the absolute value:
   (a) $|xy| = |x||y|$.
   (b) $|\frac{x}{y}| = \frac{|x|}{|y|}$ $(y \neq 0)$.
   (c) (*Triangle inequality*) $|x + y| \leq |x| + |y|$.

2. (10 pt) Let n be an integer.
   (a) Show that n is odd if and only if n^2 is odd.
   (b) Show that n is even if and only if n^2 is even.
   (c) Show that $\sqrt{2}$ is an irrational number.

3. (10 pt) Let $z \in \mathbb{C}$ be the complex number $a + bi$ with $a, b \in \mathbb{R}$, the real numbers. The *complex conjugate* of the number $z = a + bi$ is defined to be $\overline{z} = a - bi$. Prove the following.
   (a) If $z_1, z_2 \in \mathbb{C}$, then $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$.
   (b) If $z_1, z_2 \in \mathbb{C}$, then $\overline{z_1 z_2} = (\overline{z_1})(\overline{z_2})$.
   (c) If $z \in \mathbb{C}$, then $z\overline{z} \in \mathbb{R}$.
   (d) If we define the function $\phi : \mathbb{C} \longrightarrow \mathbb{R}$ by $\phi(z) = z\overline{z}$ show that for all $z_1, z_2 \in \mathbb{C}$, $\phi(z_1 z_2) = \phi(z_1)\phi(z_2)$. Is it also true that $\phi(z_1 + z_2) = \phi(z_1) + \phi(z_2)$? Prove or give a counterexample.

4. (3 pt) Prove that if n and m are positive integers, then $\frac{n+m}{2} \geq \sqrt{nm}$.

5. (3 pt) Prove that for all $\epsilon > 0$ $(\epsilon \in \mathbb{R})$, we can find a positive integer N such that $\frac{1}{N} < \epsilon$.