MATH 270
SUMMER 2004
HOMEWORK 2

Due Wednesday June 23, 2004.

1. Let x, y, z be real numbers. Prove the following.
   a) (3 pt) (*The Triangle Inequality.*) $|x + y| \leq |x| + |y|$.
   b) (3 pt) Show that if $x \leq y$ and $z < 0$ then $zx \geq zy$ and $\frac{x}{z} \geq \frac{y}{z}$.

2. (5 pt) Let $\epsilon, \alpha > 0$ be a positive real numbers. Show that there exists a natural number N such that $N\epsilon > \alpha$.

3. Let $\alpha < \beta$ be real numbers and $n \in \mathbb{N}$.
   a) (5 pt) Show that if $\beta - \alpha > n$ then there are at least n distinct integers strictly between α and β.
   b) (5 pt) Show that if $\alpha < \beta$ are real numbers then there is a rational number q such that $\alpha < q < \beta$.

4. (*Properties of Divisibility.*) Let n, m be integers with $n \neq 0$. We say that n divides m (and we write $n|m$) if there is an integer k such that $m = kn$. Prove the following properties of divisibility.
   a) (3 pt) Show that if $n|a$ and $n|b$ then $n|(ra + sb)$ for all $r, s \in \mathbb{Z}$.
   b) (3 pt) Show that if n and m are relatively prime (that is, they have no common divisors other than ± 1) and $n|am$ then $n|a$. (Hint: you may use the fact that if n and m are relatively prime, then we can find integers r and s such that $rn + sm = 1$...this is an important property of the integers that we will verify later and you will see again many times.)
   c) (3 pt) Show that if n and m are relatively prime and $n|a$ and $m|a$ then $nm|a$.

5. (5 pt) (*The Rational Root Test.*) Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial with each $a_i \in \mathbb{Z}$ $(0 \leq i \leq n)$. Suppose that this polynomial has the rational root $\frac{a}{b}$ with $a, b \in \mathbb{Z}$ having no common factors (other than ± 1). Show that $a|a_0$ and $b|a_n$. Use this to find all roots of the polynomial $6x^4 - 13x^3 - 17x^2 + 26x + 10$.