MATH 270
SUMMER 2004
HOMEWORK 4

Due Friday July 2, 2004.

1. (5 pt) Show that for all integers $n \geq 2$, $\sqrt{n} < 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}}$.

2. Recall that the Fibonacci numbers are defined recursively by $a_1 = 1 = a_2$ and $a_n = a_{n-1} + a_{n-2}$ for all $n \geq 3$.
   a) (5 pt) Show that $\gcd(a_n, a_{n+1}) = 1$ for all $n \in \mathbb{N}$.
   b) (5 pt) Show that $\gcd(a_n, a_{n+2}) = 1$ for all $n \in \mathbb{N}$.

3. (5 pt) Find a closed-form formula for

$$\int_0^{\frac{\pi}{2}} \sin^{2n}(x)dx, \; n \geq 1$$

and prove (by induction) that your formula is valid.

4. (3 pt) Prove that for every integer n, $n^3 + 6n^2 + 11n + 6$ is divisible by 3.

5. The game of poker is played with 5 cards from a standard deck (we will assume that there are no jokers).
   a) (3 pt) How many distinct 5-card hands are there?
   b) (3 pt) How many two-pair hands are there ("two-pairs" is a hand that looks like $x-x-y-y-z$ with x, y, z distinct)?
   c) (3 pt) How many flushes are there (a flush is a hand where all the cards are of the same suit)?
   d) (3 pt) How many straights are there (a straight is a consisting of 5 cards in consecutive order...we will assume that an ace can be either high or low)?
   e) (3 pt) How many straight-flushes are there (a straight-flush is a hand that is both a straight and a flush)?