1. Let \( a \in \mathbb{R} \) be positive and consider the sequence of real numbers:
\[
\{ \sqrt{a}, \sqrt{a + \sqrt{a}}, \sqrt{a + \sqrt{a + \sqrt{a}}} \ldots \}.
\]
(a) (3 pt) Show that the sequence is bounded above (that is, show that there is a real number \( N \) such that every term of the sequence is less than or equal to \( N \)).
(b) (3 pt) Show that the sequence is monotonically increasing (that is, show that the successive terms of the sequence increase in size).
(c) (3 pt) A theorem from calculus states that any bounded, monotonic sequence must converge. To what real number does the above sequence converge?

2. (5 pt) Show that for all integers \( n \geq 6 \), \( n^n > 2^n n! \).

3. (5 pt) Show that for all integers \( n \geq 2 \), \( \sqrt{n} < 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} \).

4. The Fibonacci numbers are defined recursively by the formula \( a_1 = 1, a_2 = 1 \) and \( a_n = a_{n-2} + a_{n-1} \) for all \( n \geq 3 \).
(a) (3 pt) Show that \( \gcd(a_n, a_{n+1}) = 1 \) for all \( n \in \mathbb{N} \).
(b) (3 pt) Show that \( \gcd(a_n, a_{n+2}) = 1 \) for all \( n \in \mathbb{N} \).

5. Verify the following equations for all integers \( m \geq 0 \).
(a) (5 pt) \( \int_0^{\pi/2} \cos^{2m}(x) \, dx = \frac{(2m)!}{2^{m}(m!)^2} \left( \frac{\pi}{2} \right) \).
(b) (5 pt) \( \int_0^{\pi/2} \cos^{2m+1}(x) \, dx = \frac{2^{m+1}(m!)^2}{(2m+1)!} \).

Remark: For fun, see if you can use this result to inductively find the formula of the \( n \)-dimensional sphere of radius \( R \).