# MATH 270
# SPRING 2003
# HOMEWORK 5

*Due Friday February 21, 2003.*

1. Let $a \in \mathbb{R}$ be positive and consider the sequence of real numbers:

$$\{\sqrt{a}, \sqrt{a+\sqrt{a}}, \sqrt{a+\sqrt{a+\sqrt{a}}}\cdots\}.$$

   (a) (3 pt) Show that the sequence is bounded above (that is, show that there is a real number $N$ such that every term of the sequence is less than or equal to $N$).
   (b) (3 pt) Show that the sequence is monotonically increasing (that is, show that the successive terms of the sequence increase in size).
   (c) (3 pt) A theorem from calculus states that any bounded, monotonic sequence must converge. To what real number does the above sequence converge?

2. (5 pt) Show that for all integers $n \geq 6$, $n^n > 2^n n!$.

3. (5 pt) Show that for all integers $n \geq 2$, $\sqrt{n} < 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}}$.

4. The *Fibonacci numbers* are defined recursively by the formula $a_1 = 1$, $a_2 = 1$ and $a_n = a_{n-2} + a_{n-1}$ for all $n \geq 3$.
   (a) (3 pt) Show that $\gcd(a_n, a_{n+1}) = 1$ for all $n \in \mathbb{N}$.
   (b) (3 pt) Show that $\gcd(a_n, a_{n+2}) = 1$ for all $n \in \mathbb{N}$.

5. Verify the following equations for all integers $m \geq 0$.
   (a) (5 pt) $\int_0^{\frac{\pi}{2}} cos^{2m}(x)dx = \frac{(2m)!}{2^{2m}(m!)^2}(\frac{\pi}{2})$.
   (b) (5 pt) $\int_0^{\frac{\pi}{2}} cos^{2m+1}(x)dx = \frac{2^{2m}(m!)^2}{(2m+1)!}$.

*Remark*: For fun, see if you can use this result to inductively find the formula of the $n$-dimensional sphere of radius $R$.