MATH 270
SUMMER 2007
HOMEWORK 5

Due Friday June 29, 2007.

1. (5pt) Find a formula for

$$\int_0^{\frac{\pi}{2}} \cos^n(x)dx, n \in \mathbb{N}_0$$

in terms of n and prove that your formula works (hint: it might be helpful to consider the case where n is odd and the case where n is even). For extra credit, use this to find the volume of an $n-$dimensional sphere of radius R.

2. Verify the following.
   a) (5 pt) For all integers $n \geq 2$, $\sqrt{n} < 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}}$.
   b) (5 pt) For all integers $n \geq 6$, $n^n > 2^n n!$.

3. The *Fibonacci numbers* are defined recursively by the formula, $a_1 = a_2 = 1$ and $a_n = a_{n-2} + a_{n-1}$ for all $n \geq 2$. Verify the following properties of the Fibonacci numbers.
   a) (5 pt) $\gcd(a_n, a_{n+1}) = 1$ for all $n \in \mathbb{N}$.
   b) (5 pt) $\gcd(a_n, a_{n+2}) = 1$ for all $n \in \mathbb{N}$.
   c) (5 pt) $\alpha^{n-1} \geq a_n$ for all $n \in \mathbb{N}$ where $\alpha = \frac{1+\sqrt{5}}{2}$ (hint: it might be useful to note that α is a root of the polynomial $x^2 - x - 1$).

4. Suppose that you are playing a game of straight poker with a standard deck of 52 cards.
   a) (3 pt) How many distinct 5 card hands are there?
   b) (3 pt) How many three of a kind hands are there (a hand of the form $x - x - x - y - z$ with x, y, z distinct)?
   c) (3 pt) How many flushes are there (5 cards of the same suit)?
   d) (3 pt) How many straights are there (5 consecutive cards...the ace can be played high or low)?
   e) (3 pt) How many three of a kind hands are there if you introduce a single extra wild card?
   f) (3 pt) How many two pair hands (a hand of the form $x - x - y - y - z$ with x, y, z distinct) are there in the deck with the single extra wild card?
   g) (3 pt) Explain why you should never bet on a two-pair hand if there is even one wild card in the deck.

5. (3 pt) Let $r \leq n$ be natural numbers. Prove that the binomial coefficient

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

is a natural number.