MATH 270
SUMMER 2007
HOMEWORK 6

Due Monday July 9, 2007.

1. Define the relation $\sim$ on $\mathbb{R}$ by $x \sim y$ if and only if $x - y \in \mathbb{Z}$.

   a) (5 pt) Show that $\sim$ defines an equivalence relation on $\mathbb{R}$.

   b) (5 pt) Describe the equivalence classes of $\mathbb{R}/\sim$.

2. Consider the collection of (real) vectors in the plane (that is, consider the vector space $\mathbb{R}^2$).

   a) (3 pt) We first declare that $\vec{v} \sim \vec{w}$ if and only if $\|\vec{v}\| = \|\vec{w}\|$. Is $\sim$ an equivalence relation on $\mathbb{R}^2$? If so, describe the equivalence classes.

   b) (3 pt) Now we declare that $\vec{v} \approx \vec{w}$ if and only if $\vec{v} = \lambda\vec{w}$ for some nonzero $\lambda \in \mathbb{R}$. Is this an equivalence relation on $\mathbb{R}^2$? If so, describe the equivalence classes.

   c) (3 pt) Carefully describe how your answer would change to part b) if we allowed λ to be *any* real number.

3. (5 pt) Let $n \in \mathbb{N}\bigcup\{0\}$. We define the equivalence relation $\sim_n$ on $\mathbb{Z}$ by declaring that $a \sim_n b$ if and only if $a - b$ is a multiple of n. Show that this is an equivalence relation and that the number of equivalence classes of $\mathbb{Z}/\sim_n$ is given by

$$|\mathbb{Z}/\sim_n| = \begin{cases} n, & \text{if } n \geq 1; \\ \infty, & \text{if } n = 0. \end{cases}$$

4. (3 pt) Let S be a nonempty set. Show that there is a unique partial ordering on S that is also an equivalence relation. Carefully describe this partial ordering/equivalence relation.

5. (5 pt) In your book, a well-ordered set is defined to be a totally ordered set with the property that every subset has a least element. Show that this definition is slightly redundant by showing that any partially ordered set with the property that every subset has a least element is automatically totally ordered.

6. (3 pt) The real numbers $\mathbb{R}$ is a totally ordered set under the standard notion of $\leq$. Prove that $(\mathbb{R}, \leq)$ is not well-ordered.