MATH 270
SPRING 2003
HOMEWORK 8

Due Friday April 11, 2003.

1. (*All intervals in $\mathbb{R}$ are of equal size.*) Let (a, b) and (c, d) be open intervals (with $a \neq b$ and $c \neq d$).

   a) (5 pt) Show that $|(a, b)| = |(c, d)|$.
   b) (5 pt) Show that $|[a, b]| = |(a, b)|$.
   c) (5 pt) Show that $|(a, b)| = |\mathbb{R}|$.

2. (5 pt) (*Another concrete example of a "bigger infinity"*) Consider the interval $[0, 1]$ and let $F_{[0,1]}$ be the set of all functions from $[0, 1]$ to $[0, 1]$. Show that $|[0, 1]| < |F_{[0,1]}|$.

3. (3 pt) Let $T_n = \{0, 1, 2, \cdots, n-1\}$ $(n > 1)$ and let D_n be the set of infinite sequences $\{t_k\}_{k=1}^{\infty}$. Show that $|D_n| = |\mathbb{R}|$. Why did I demand $n > 1$ in this problem?

4. (*The Cantor middle thirds set.*) Consider the interval $I = [0, 1]$. We define a subset of I recursively. In the first step we consider the interval obtained by removing the middle third of $[0, 1]$, (which is $(\frac{1}{3}, \frac{2}{3})$), and we let $I_1 = [0, 1] \setminus (\frac{1}{3}, \frac{2}{3}) = [0, \frac{1}{3}] \bigcup [\frac{2}{3}, 1]$. In the next step we remove the middle third of each of the two pieces. That is, $I_2 = I_1 \setminus ((\frac{1}{9}, \frac{2}{9}) \bigcup (\frac{7}{9}, \frac{8}{9}))$. In the step after this we again remove the middle third of the remaining pieces and we continue this process indefinitely. Let C be the subset of $[0, 1]$ that is left over.

   a) (3 pt) Show that the total "length" of what is removed is 1 (the same as the length of $[0, 1]$!).
   b) (3 pt) Show that $|C| = |[0, 1]|$. (So despite the fact that you have removed intervals whose total length is 1, you are left over with as many points as there are real numbers!)