MATH 270
SUMMER 2004
HOMEWORK 8

Due Monday July 26, 2004.

1. Let G be a group. Prove the following.

   a) (3 pt) If $x \in G$, then x^{-1} is unique.
   b) (3 pt) If $x \in G$ then $(x^{-1})^{-1} = x$.
   c) (3 pt) If $x, y \in G$ then $(xy)^{-1} = y^{-1}x^{-1}$.

2. Let G be a finite group with subgroup $H \subseteq G$ and $x \in G$ an element. We define the order of H (written $|H|$) to be the cardinality of the set H. Additionally, we define the order of x (written $|x|$) to be the smallest positive integer such that $x^n = e$ (and if no such positive integer exists, we say that $|x| = \infty$). Consider the set $\langle x \rangle = \{x^n | n \in \mathbb{Z}\}$.

   a) (5 pt) Show that $\langle x \rangle$ is a subgroup of G.
   b) (5 pt) Show that if $|x|$ is finite then $|x| = |\langle x \rangle|$.
   c) (5 pt) Show that if $|x| = n < \infty$ and $x^m = e$ for some $m \in \mathbb{N}$ then n divides m.
   d) (5 pt) Show that if $x^m = e$ and $x^n = e$ for some $n, m \in \mathbb{N}$ then $|x|$ divides $\gcd(n, m)$.

3. (5 pt) Show that if G is a group with the property that $x^2 = e$ for all $x \in G$, then G is abelian.

4. (5 pt) Show that if G is a finite group and $x \in G$ then $|x|$ is finite.

5. (3 pt) Let G and H be groups. We define $G \times H = \{(g, h) | g \in G, h \in H\}$ with a binary operation given by

$$(g_1, h_1)(g_2, h_2) = (g_1 g_2, h_1 h_2).$$

Show that this operation makes the set $G \times H$ into a group (called the direct product of G and H).