MATH 270
SUMMER 2007
HOMEWORK 8

Due Friday July 20, 2007.

1. Let $a < b$ and $c < d$ all be real numbers.
   a) (3 pt) Show that $|(a,b)| = |(c,d)|$.
   b) (3 pt) Show that $|(a,b)| = |\mathbb{R}|$.

2. Let $a < b$ and $c < d$ be real numbers and A, B, C, D be sets.
   a) (5 pt) Show that if $|A| = |B|$ and $|C| = |D|$ then $|A \times C| = |B \times D|$.
   b) (5 pt) Show that $|(a,b) \times (c,d)| = |(0,1)|$.
   c) (5 pt) Show that $|\mathbb{R} \times \mathbb{R}| = |\mathbb{R}|$.

3. Let A and B be nonempty sets.
   a) (5 pt) Let $F_A = \{f | \ f : A \longrightarrow A\}$ be the set of functions from A to A. Show that $|A| < |F_A|$.
   b) (5 pt) If A and B are finite sets, count the number of functions from A to B. How many are one to one and how many are onto?

See if you can show that there are

$$\sum_{k=0}^{m-1} (-1)^k \binom{m}{k} (m-k)^n$$

functions from a set of n elements onto a set of m elements with $m \leq n$. Note that this has the corollary that $\sum_{k=0}^{n-1}(-1)^k \binom{n}{k}(n-k)^n = n!$.

4. Consider the set, I, of all integrable functions on the interval $[-1,1]$. We say that two functions in I, f and g, are equivalent ($f \sim g$) if $\int_{-1}^{1}(f(x) - g(x))\, dx = 0$.
   a) (3 pt) Show that $\sim$ defines an equivalence relation on I.
   b) (3 pt) Find a representative for each equivalence class in $I/\sim$.
   c) (3 pt) Show that $|I/\sim| = |\mathbb{R}|$.

5. (*The Cantor middle thirds set.*) Consider the interval $I = [0,1]$. We define a subset of I recursively. In the first step we consider the interval obtained by removing the middle third of $[0,1]$, (which is $(\frac{1}{3}, \frac{2}{3})$), and we let $I_1 = [0,1] \setminus (\frac{1}{3}, \frac{2}{3}) = [0, \frac{1}{3}] \bigcup [\frac{2}{3}, 1]$. In the next step we remove the middle third of each of the two pieces. That is, $I_2 = I_1 \setminus ((\frac{1}{9}, \frac{2}{9}) \bigcup (\frac{7}{9}, \frac{8}{9}))$. In the step after this we again remove the middle third of the remaining pieces and we continue this process indefinitely. Let C be the subset of $[0,1]$ that is left over.
   a) (3 pt) Show that the total "length" of what is removed is 1 (the same as the length of $[0,1]$!).
   b) (3 pt) Show that $|C| = |[0,1]|$. (So despite the fact that you have removed intervals whose total length is 1, you are left over with as many points as there are real numbers!)