MATH 270
SPRING 2003
EXAM 2
TAKE HOME PORTION

Due Wednesday April 2, 2003.

Instructions: You may use notes and your book, but I am the only biological resource that you should utilize. This portion of the exam will be weighted equally with the in class portion.

1. The real numbers $\mathbb{R}$ is called an *ordered field* under the standard notion of $\leq$. By this we mean that $\mathbb{R}$ is a totally ordered set, and additionally, the following two axioms are satisfied.

i) If $x, y, z \in \mathbb{R}$ and $x \leq y$ then $x + z \leq y + z$.
ii) If $x, y \in \mathbb{R}$ are such that $0 < x$ and $0 < y$ then $0 < xy$.

Let $x, y, z \in \mathbb{R}$ and prove the following.

a) (3 pt) $x > 0 \Longleftrightarrow -x < 0$.
b) (3 pt) If $x > 0$ and $y < z$ then $xy < xz$.
c) (3 pt) If $x < 0$ and $y < z$ then $xy > xz$.
d) (3 pt) $x^2 \geq 0$ and equality holds if and only if $x = 0$.
e) (3 pt) If $0 < x < y$ then $0 < \frac{1}{y} < \frac{1}{x}$.

2. (10 pt) Recall that the complex numbers are given by $\mathbb{C} = \{x+iy | x, y \in \mathbb{R}\}$. Show that the complex numbers is not an ordered field (that is, show that there is no total ordering on $\mathbb{C}$ satifying the two extra axioms i) and ii) from problem 1).

3. Let $n, m \in \mathbb{N} \bigcup \{0\}$ be nonnegative integers. We will denote the set of equivalence classes modulo n (respectively m) be denoted by $\mathbb{Z}_n$ (respectively $\mathbb{Z}_m$). We also note the canonical projection from $\mathbb{Z}$ to $\mathbb{Z}_n$ by π_n (where $\pi_n(k) = [k]_n$, the equivalence class of k modulo n).

a) (3 pt) Find the preimage (under π_n) of the element $[k]_n$.
b) (3 pt) Show that π_n is onto.
c) (3 pt) Show that there is a function $f_{n,m} : \mathbb{Z}_n \longrightarrow \mathbb{Z}_m$ such that $f_{n,m} \circ \pi_n = \pi_m$ if and only if n is a multiple of m.
d) (3 pt) Show that the function from part c) (if it exists) is necessarily onto.
e) (3 pt) Show that $f_{n,m}$ is one to one if and only if $n = m$.

4. (10 pt) Give an example of a function from $\mathbb{Z}$ to $\mathbb{N}$ that is one to one and onto or explain why this cannot be done. Give an example of a function from $\mathbb{N}$ to $\mathbb{Q}$ that is one to one and an example of a function from the positive rational numbers to the natural numbers that is one to one or explain why this cannot be done.