MATH 270
SPRING 2003
FINAL EXAM

1. (10 pt) Let S be a finite set with $n \geq 2$ elements. Show that the number of subsets of order 2 of S is given by

$$\frac{n(n-1)}{2}.$$

2. (5 pt) Give an example of functions:

$$A \xrightarrow{g} A \xrightarrow{f} A$$

with f not one to one and g not onto such that $f \circ g$ is a bijection.

3. (10 pt) Consider the set S of all rearrangements on the Rubik's cube (obtained by rotations). We define a relation $\sim$ on S be declaring that two arrangements are similar ($R_1 \sim R_2$) if when looking at only the corners of the cube (with the white side forward) the two arrangements are indistinguishable. Show that $\sim$ defines an equivalence relation on S.

4. Let G be a group and $x, y \in G$ two elements such that $xy = yx$. Also assume that the order of x is n and the order of y is m (both finite). (For this problem you may freely use the fact that $\gcd(n, m)\text{lcm}(n, m) = nm$ for positive integers n, m.)

a) (3 pt) Let k be a positive integer. Show that the order of x^k is $\frac{n}{\gcd(k,n)}$.
b) (3 pt) Show that if $\gcd(n, m) = 1$ and $x^k = y^j$ then $x^k = y^j = e$ (hint: if $x^k = y^j$ then the group generated by x^k is contained in the group generated by y).
c) (3 pt) Show that the order of xy divides $\text{lcm}(n, m)$.
d) (3 pt) Use part b) to show that if $\gcd(n, m) = 1$ then the order of xy is nm.

5. Let G be a group (with operation $\circ$). We define a new operation on the set G by $x \star y = y \circ x$.

a) (5 pt) Show that G with the operation $\star$ forms a group.
b) (5 pt) Show that the map $f : (G, \circ) \longrightarrow (G, \star)$ given by $f(x) = x^{-1}$ is an isomorphism of groups.

6. Assume that the Rubik's cube is scrambled using $D'R'$.

a) (3 pt) Compute the cycle decomposition for the corners.
b) (3 pt) Compute the cycle decomposition for the sides.
c) (3 pt) How many times must we do the move $D'R'$ before all the blocks come back into their correct positions?
d) (3 pt) Let m be the answer to c). It turns out that after doing the move $D'R'$ m times the sides are all oriented correctly, but the corners are not. How many times must we do the move $D'R'$ before the cube returns to its original state?
e) (3 pt) What would the answer to d) have been if some of the sides were not oriented correctly after doing $D'R'$ m times?